PoG Set 8 answers

1. i) $\frac{h\nu}{kT} << 1$.This is the Rayleigh-Jeans approximation. The lowest temperature in the disc occurs at $r_{\max}$. For frequencies such that $h\nu << kT(r_{\max})$, therefore, we have

$$L_\nu \propto \nu^3 \int_{r_{lso}}^{r_{\max}} \frac{rdr}{e^{h\nu/kT(r)}-1} \approx \nu^3 \int_{r_{lso}}^{r_{\max}} \frac{rdr}{1+h\nu/kT(r)-1} \approx \nu^3 \frac{1}{h\nu} k \int_{r_{lso}}^{r_{\max}} T(r)\, rdr \propto \nu^2,$$ [3marks]

ii) $L_\nu \propto \nu^3 \int_{r_{lso}}^{r_{\max}} \frac{rdr}{e^{h\nu/kT(r)}-1}$, we write $T(r) = \left(\frac{GM\dot{m}}{8\pi r^3 \sigma}\right)^{1/4} = T(r_{lso})\left(\frac{r_{lso}}{r}\right)^{3/4} = T_* r^{-3/4}$. Introduce new notation $x = h\nu/kT(r) = h\nu/(kT_* r^{-3/4})$. Thus, $x \propto r^{3/4}\nu$ or $r \propto x^{4/3}\nu^{-4/3}$ and $dr \propto \nu^{-4/3}(4/3)x^{1/3}dx$. Finally, $L_\nu \propto \nu^3 \nu^{-4/3}\nu^{-4/3} \int_{r_{lso}}^{r_{\max}} \frac{x^{4/3}(4/3)x^{1/3}dx}{e^x-1}$. The integral is just a number thus

$L_\nu \propto \nu^{1/3}$.

[3marks]

iii) $\frac{h\nu}{kT} >> 1$ At high frequencies, we have $\frac{1}{e^{h\nu/kT}-1} \approx e^{-h\nu/kT}$; The highest temperature in the disc occurs at r_{lso}. For frequencies such that $h\nu >> kT(r_{lso})$, therefore, we have

$$L_\nu \propto \nu^3 \int_{r_{lso}}^{r_{\max}} e^{-h\nu/kT(r)} rdr \approx \nu^3 e^{-h\nu/kT(r_{lso})} \int_{r_{lso}}^{r_{\max}} rdr \approx \nu^3 (r_{\max}^2/2) e^{-h\nu/kT(r_{lso})} \propto \nu^3 e^{-h\nu/kT(r_{lso})}.$$

Here , in the second approximation, we have replaced the exponential term with its maximum value and, in the third, have again assumed that $r_{\max} >> r_{lso}$ (i.e. we ignore lower bond of integration).

[3marks]

[3+3+3=9 marks]

2.From special relativity we have

$$\gamma = \frac{E}{m_e c^2} \quad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}}. \quad (2.1)$$ [1 mark]

Hence $$\beta = \sqrt{1-\frac{1}{\gamma^2}} \quad (2.2)$$ [1 mark]

If γ is large, then $1/\gamma << 1$ so that we may expand the square root to lowest order in the small quantity $1/\gamma^2$

$$\beta = \left(1-\frac{1}{\gamma^2}\right)^{1/2} \approx 1-\frac{1}{2\gamma^2} \quad \gamma >> 1 \quad (2.3)$$ [3 marks]

[1+1+3=5marks]

Using $m_e c^2 = 0.511$ MeV we construct the table:

E (MeV)	γ eq. (1.1)	β eq. (1.2)	$1-\beta=\frac{1}{2\gamma^2}$ eq. (1.3)
10	19.57	0.9986	1.3×10^{-3}
100	195.69	0.999987	1.3×10^{-5}
1000	1956.95	0.9999987	1.3×10^{-7}
10000	19569.47	0.9999999….	1.3×10^{-9}

[5 marks]
It is clear that for all the listed energies it is a perfectly adequate approximation to assume $\beta = 1$ unless we have a good reason to require greater accuracy and that is usually not the case in astrophysics! (Even at *E = 1 MeV,* γ = 1.96 gives $\beta = 0.860$ using equation (2.2) and $\beta \approx 0.869$ using the approximate equation (2.3).) **[2 marks]**