Physics of Galaxies

ANSWERS: SET NUMBER 4

1. For a star with mass m_1 moving under the strong gravitational influence of the one with mass m_2 :

$$\frac{Gm_1m_2}{b_{strong}} = \frac{m_1v^2}{2} \Rightarrow b_{strong} = \frac{2Gm_2}{v^2}$$ **[2 marks]**

2. First take time derivative of the linearised continuity equation -- the logic here is that we see that the desired equation contains 2nd order time derivative of ρ_1:

$$\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \nabla \cdot \frac{\partial \vec{V}_1}{\partial t} = 0; \qquad (1)$$

Then take divergence of the linearised equation of motion -- as we see that 2nd term in Eq.(1) has the desired structure:

$$\nabla \cdot \frac{\partial \vec{V}_1}{\partial t} = \nabla \cdot \left(-\frac{\nabla p_1}{\rho_0} - \nabla \varphi_1 \right) = -\frac{u^2}{\rho_0}\Delta\rho_1 - \Delta\varphi_1 = -\frac{u^2}{\rho_0}\Delta\rho_1 - 4\pi G\rho_1; \qquad (2)$$

where we have used equations: $\Delta\varphi_1 = 4\pi G\rho_1$ and $\nabla p_1 = u^2\nabla\rho_1$.

Multiplying Eq.(2) by ρ_0 and substituting into Eq.(1) yields the desired equation $\frac{\partial^2 \rho_1}{\partial t^2} - 4\pi G\rho_0\rho_1 - u^2\Delta\rho_1 = 0$ **[8 marks]**. Fourier ansatz $f = \tilde{f}e^{i(\omega t - kr)}$ gives $\frac{\partial^2 \tilde{\rho}_1}{\partial t^2} = -\omega^2\tilde{\rho}_1$ and $\Delta\tilde{\rho}_1 = \frac{\partial^2 \tilde{\rho}_1}{\partial r^2} = -k^2\tilde{\rho}_1$. Thus, we have: $-\omega^2\tilde{\rho}_1 - 4\pi G\rho_0\tilde{\rho}_1 + u^2k^2\tilde{\rho}_1 = 0$. Canceling $\tilde{\rho}_1$ and rearranging yields: $\omega^2 = u^2k^2 - 4\pi G\rho_0$ **[2 marks]**.

3. a) If $k < \left(\frac{4\pi G\rho}{u^2}\right)^{1/2}$, then we have from the dispersion relation, $\omega^2 < 0$, so that ω is imaginary which leads to the Jeans instability i.e. exponentially growing solutions. This occurs for the case of Jeans length of $\lambda > \lambda_J = \left(\frac{\pi u^2}{G\rho}\right)^{1/2}$ **[4 marks].**

b) Jeans mass can be obtained from $M = (4\pi/3)R^3\rho$ [1 mark]. If the diameter of the collapsing cloud is $l=2R$, one needs to realize that for the possible modes of oscillation, cloud edges need to be nodes of the standing wave because outside the cloud no oscillation can be sustained [1 mark]. A trough or crest of the wave (bounded by cloud edges) is then half the wavelength. Therefore, $l = \lambda_J/2$ and $R = \lambda_J/4$ [2 marks]. Thus,

$$M_J = (4\pi/3)(\lambda_J/4)^3\rho = \frac{4\pi}{3\cdot 4^3}\left(\frac{\pi u^2}{G\rho}\right)^{3/2}\rho = \frac{\pi^{5/2}}{48}\frac{u^3}{G^{3/2}}\frac{1}{\sqrt{\rho}}$$ [2 marks]. **[1+1+2+2=6 marks].**

c) When thermal effects are neglected the dispersion relation reads as $\omega^2 = -4\pi G\rho$. This means that $\alpha = \sqrt{4\pi G\rho}$. If a physical quantity decays (reduces) in time e-times of its original value, where e is the base of natural logarithm, it is said that it is reduced to zero. Conversely, the timescale over which density goes up by a factor of e, can be used as an indication of a full collapse (in the absence of thermal pressure, which normally opposes the gravitational collapse). Thus, $\tau_{ff} = 1/\alpha = 1/\sqrt{4\pi G\rho} \propto 1/\sqrt{G\rho}$ [3 marks] is the time it takes a cloud to collapse in the absence of supporting pressure [2 marks]. **[3+2=5 marks]**

[Total marks available 27]