UNIVERSITY OF LONDON
(University College London)

B.Sc. DEGREE 1999

PHYSICS B221: Mathematical Methods in Physics

Credit will be given for all work done. [For guidance, a student should aim to answer the equivalent of FOUR complete questions in the time available.]

N.B. The Kronecker delta symbol for two integers m and n is defined by

$$\delta_{mn} = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \,. \end{cases}$$

1. By writing both sides in terms of Cartesian coordinates, or otherwise, verify the identities

$$\vec{\nabla} \cdot (\chi \vec{\nabla} \psi) = \vec{\nabla} \chi \cdot \vec{\nabla} \psi + \chi \nabla^2 \psi \,,$$ **[4 marks]**

$$\vec{\nabla} \times (\psi \vec{A}) = \vec{\nabla} \psi \times \vec{A} + \psi (\vec{\nabla} \times \vec{A}) \,,$$ **[4 marks]**

where χ and ψ are scalar functions and $\vec{A}$ a vector function.

In spherical polar coordinates ($x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$), the line element is given by

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \,,$$

where $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ are basis vectors in the directions of increasing r, θ and ϕ respectively. Show that in these coordinates

$$\vec{\nabla}\psi = \frac{\partial \psi}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\,\hat{\phi} \,.$$ **[3 marks]**

If $\psi = z^2$, evaluate $\vec{\nabla}\psi$ in both Cartesian and spherical polar coordinates and show that they are equal. **[5 marks]**

For the field $\vec{A} = r^n\,\vec{r}$, evaluate $\vec{\nabla} \cdot \vec{A}$. **[4 marks]**

2. The divergence theorem states that, for any vector field $\vec{A}$,

$$\int_V \vec{\nabla} \cdot \vec{A}\, dV = \int_S \vec{A} \cdot \hat{n}\, dS\ ,$$

where S is the closed surface surrounding the volume V and $\hat{n}$ is a unit vector directed along the outward normal to S.

In Cartesian coordinates,

$$\vec{A} = xy\hat{\imath} + y\hat{\jmath} + z^3\hat{k}\ .$$

Find the divergence $\vec{\nabla} \cdot \vec{A}$ of this field. **[4 marks]**

If the volume V is that of a right cylinder defined by $x^2 + y^2 \leq 1$ and $-1 \leq z \leq +1$, derive the volume integral of the divergence of $\vec{A}$. **[6 marks]**

Show that the integral of $\vec{A} \cdot \hat{n}$ over the two end caps equals 2π, and that over the curved surface is also equal to 2π. **[4 marks]** **[5 marks]**

Hence verify the divergence theorem in this case. **[1 mark]**

You may assume that in cylindrical polar coordinates the volume element can be written as

$$dV = r\, dr\, d\theta\, dz\ .$$

3. Define a Hermitian matrix. **[2 marks]**

Show that the eigenvalues of a Hermitian matrix are real and that the eigenvectors corresponding to different eigenvalues are orthogonal. **[6 marks]**

Show that the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -5 & 1 \\ -5 & 11 & -5 \\ 1 & -5 & 5 \end{pmatrix}$$

has eigenvalues 1, 4, and 16. **[4 marks]**

Find the eigenvectors corresponding to $\lambda = 1$ and $\lambda = 4$ and show that they are indeed orthogonal. **[6 marks]**

Verify in this case that the trace of the matrix is equal to the sum of the eigenvalues. **[2 marks]**

4. Laplace's equation in plane polar coordinates is

$$r^2 \frac{\partial^2 F}{\partial r^2} + r \frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial \theta^2} = 0 \, .$$

Assuming that the function F can be separated as

$$F(r, \theta) = R(r) \times \Theta(\theta) \, ,$$

derive ordinary differential equations for $R(r)$ and $\Theta(\theta)$. **[5 marks]**

Solve the radial equation by trying for a solution of the form $R(r) = r^\alpha$. Hence show that the most general single-valued solution may be written as

$$F(r, \theta) = \sum_{n=0}^{\infty} \left\{ A_n r^n + B_n r^{-n} \right\} (C_n \cos n\theta + D_n \sin n\theta) \, .$$ **[10 marks]**

Find the form of the solution for which $F(r, \theta) = F_1 \cos \theta$ when $r = a$, where F_1 is a constant. Is the solution unique? **[5 marks]**

5. Show that the second order differential equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + b^2 y = 0$$

has two solutions of the form

$$y = \sum_{\lambda=0}^{\infty} a_\lambda \, x^{\lambda + k} \, , \qquad a_0 \neq 0$$

with $k = 0$ or $k = 1$. **[6 marks]**

Find the ratio $a_{\lambda+2}/a_\lambda$ in both series. **[4 marks]**

For the special case of $b = m$ (m a positive integer) show that the series expansion for one of the solutions terminates at $\lambda = m - k$. **[4 marks]**

Verify up to $m = 2$ that the resulting polynomial may be written as $y = C_m \cos m\theta$, where $\cos \theta = x$ and C_m is a constant. **[6 marks]**

6. A function $f(x)$, which is periodic with period 2π, is expanded in a Fourier series

$$f(x) = \tfrac{1}{2}A_0 + \sum_{m=1}^{\infty} A_m \cos mx + \sum_{m=1}^{\infty} B_m \sin mx$$

Show that the coefficients are given by

$$A_m = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,\cos mx\,dx\,, \qquad B_m = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,\sin mx\,dx\,.$$ **[5 marks]**

If, further,

$$\begin{aligned} f(x) &= x\,, & 0 \le x \le \pi\,, \\ &= -x\,, & -\pi \le x \le 0\,, \end{aligned}$$

sketch the function in the range $-4\pi \le x \le 4\pi$. **[2 marks]**

Show that all the B_m coefficients in the Fourier expansion vanish and that

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m\ \mathrm{odd}} \frac{1}{m^2}\cos mx\,.$$ **[8 marks]**

Hence deduce that

$$\sum_{m\ \mathrm{odd}} \frac{1}{m^2} = \frac{\pi^2}{8}\,.$$ **[2 marks]**

Show that the error made by only taking the first three terms in the sum is less than 7%. **[2 marks]**

You may use the integrals

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \pi\,\delta_{m\,n}\,,$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \pi\,\delta_{m\,n}\,,$$

$$\int x\cos mx\,dx = \frac{x}{m}\sin mx + \frac{1}{m^2}\cos mx + C\,.$$

7. The Legendre polynomials $P_\ell(x)$ may be defined by the Rodrigues formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{d}{dx}\right)^\ell (x^2 - 1)^\ell \, .$$

Hence find explicit expressions for $P_0(x)$, $P_1(x)$, and $P_2(x)$. **[4 marks]**

Show that these <u>three</u> polynomials satisfy the orthonormality relation

$$\int_{-1}^{+1} P_\ell(x)\, P_n(x)\, dx = \frac{2}{2\ell + 1}\, \delta_{\ell n} \, ,$$ **[5 marks]**

Use Leibniz's rule for differentiating a product,

$$\left(\frac{d}{dx}\right)^\ell [f(x)\, g(x)] = \sum_{n=0}^{\ell} \frac{\ell!}{(\ell - n)!\, n!} \left[\left(\frac{d}{dx}\right)^n f(x)\right] \left[\left(\frac{d}{dx}\right)^{\ell - n} g(x)\right] \, ,$$

to show that the Rodrigues formula implies that $P_\ell(1) = 1$ for all values of ℓ. **[5 marks]**

Derive an integral expression for the $Q_\ell(z)$ in the expansion

$$\frac{1}{z - x} = \sum_{\ell=0}^{\infty} Q_\ell(z)\, P_\ell(x) \, ,$$

where $z > 1$. **[3 marks]**

Find explicit expressions for the $Q_\ell(x)$ when $\ell = 0$ and $\ell = 1$. **[3 marks]**

END OF PAPER