PHAS1245: Problem Sheet 7 - Solutions

1. Easier, but not necessary, to take the log of the given relation first:

$$\ln p + \ln V = \ln R + \ln T - \frac{a}{VRT}\,.$$

Then we do implicit differentiation to obtain the tree partial derivatives. First wrt V with constant T:

$$\frac{1}{p}\left(\frac{\partial p}{\partial V}\right)_T + \frac{1}{V} = \frac{a}{V^2RT} \Rightarrow \left(\frac{\partial p}{\partial V}\right)_T = \frac{p(a-VRT)}{V^2RT}\,,$$

then wrt T with constant p:

$$\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = \frac{1}{T} - \frac{a}{R}\left(-\frac{1}{V^2} - \frac{1}{VT^2}\right) \Rightarrow \left(\frac{\partial V}{\partial T}\right)_p = -\frac{V(a+VRT)}{T(a-VRT)}\,,$$

and finally wrt p with V constant:

$$\frac{1}{p} = \frac{1}{T}\left(\frac{\partial T}{\partial p}\right)_V + \frac{a}{VRT^2}\left(\frac{\partial T}{\partial p}\right)_V \Rightarrow \left(\frac{\partial T}{\partial p}\right)_V = \frac{VRT^2}{p(a+VRT)}\,.$$

Muitlplying the three derivatives we readily find that their product is -1.

2. For the LHS, we have

$$\frac{\partial^2 f}{\partial x^2} = 2y^2\,, \qquad \frac{\partial^2 f}{\partial y^2} = 2x^2 \qquad \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2(x^2+y^2) = 2\rho^2\,.$$

In polar coordinates $f(\rho,\phi) = \rho^4\sin^2\phi\cos^2\phi = (\rho^4\sin^2 2\phi)/4$. Hence

$$\frac{\partial f}{\partial \rho} = \rho^3\sin^2 2\phi \qquad \frac{\partial^2 f}{\partial \rho^2} = 3\rho^2\sin^2 2\phi$$

$$\frac{\partial^2 f}{\partial \phi^2} = \frac{\partial}{\partial \phi}(\rho^4\sin 2\phi\cos 2\phi) = \rho^4(2\cos^2 2\phi - 2\sin^2 2\phi)\,,$$

and the RHS becomes

$$3\rho^2 + \frac{1}{\rho}\rho^3\sin^2 2\phi + \frac{1}{\rho^2}\rho^4(2\cos^2 2\phi - 2\sin^2 2\phi) = \rho^2(2\cos^2 2\phi + 2\sin^2 2\phi)\,,$$

which becomes $2\rho^2$, equal to the LHS result.

3. We have

$$\frac{\partial f}{\partial y} = -12x + 2by = 0 \Rightarrow y = \frac{6x}{b}$$

$$\frac{\partial f}{\partial x} = 3x^2 - 12y + 48 = 0 \Rightarrow x^2 - \frac{24}{b}x + 16 = 0\,.$$

This quadratic equation has two, one or zero real solutions depending on whether $\Delta = (24/b)^2 - 64 = 64(9/b^2 - 1)$ is positive, 0 or negative. So, (a) two solutions if $|b| < 3$, (b) one solution if $|b| = 3$ and (c) no real solutions if $|b| > 3$.

When $|b| < 3$, the two solutions for x are

$$x_+ = \frac{12 + 4\sqrt{9 - b^2}}{b} = \frac{d_+}{b} \qquad x_- = \frac{12 - 4\sqrt{9 - b^2}}{b} = \frac{d_-}{b}$$

and correspondingly

$$y_+ = \frac{6x_+}{b} = \frac{6d_+}{b^2} \qquad y_+ = \frac{6x_-}{b} = \frac{6d_-}{b^2}\,.$$

To determine the nature of these points we need

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6x\,,\ \ f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2b\,,\ \ f_{xx}f_{yy} = 12bx\,,\ \ \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = (-12)^2 = 144\,.$$

For the point (x_+, y_+), $f_{xx}f_{yy} = 12d_+$. The smallest value of d_+ is when $|b| \to 3$ and then $d_+ \to 12$, but it is always greater than 12. Therefore $f_{xx}f_{yy} > 144 (= f_{xy}^2)$, $f_{xx} = 6x_+ = 6d_+/b$ and $f_{yy} = 2b$. So, if $-3 < b < 0$ this poit is a maximum and if $0 < b < 3$ the point is a minimum.

For the point (x_-, y_-), $f_{xx}f_{yy} = 12d_-$. The largest value of d_- is when $|b| \to 3$ and then $d_- \to 12$, but it is always smaller than 12. Therefore $f_{xx}f_{yy} < 144 (= f_{xy}^2)$ and this point is a saddle point.

4. The problem here is to maximise the volume of the rectangular parallelipiped $f = 8xyz$ subject to the ellipsoidal constraint equation ϕ. We have

$$F(x, y, z) = f + \lambda\phi = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\,.$$

The three partial derivatives of F are now set to zero. We find

$$8yz + \lambda\frac{2x}{a^2} = 0\,,\quad 8xz + \lambda\frac{2y}{b^2} = 0\,, \text{and}\quad 8xy + \lambda\frac{2z}{c^2} = 0\,.$$

From these we readily find

$$3 \times 8xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0 \Rightarrow 24xyz + 2\lambda = 0 \Rightarrow \lambda = -12xy\,.$$

Putting this back into the partial derivative equations which were set to zero we find

$$x^2 = \frac{1}{3}a^2\ \ ,\ \ y^2 = \frac{1}{3}b^2\ \ ,\ \ z^2 = \frac{1}{3}c^2\ \ \text{and}\ \ 8xyz = \frac{8abc}{3\sqrt{3}}\,.$$