PHAS1245: Problem Sheet 6 - Solutions

1. The condition for the two lines to be parallel is that the vectors $\vec{b}_1$ and $\vec{b}_2$ are parallel, i.e. $\vec{b}_1 \times \vec{b}_2 = 0$.

   To find the distance between them: if P and Q are the points where a line segment perpendicular to the two lines crosses them, such that $\vec{PQ} = \vec{d}$, where $\vec{d}$ the distance we are after, we can find a value for each of λ_1 and λ_2 such that

$$\vec{a}_1 + \lambda_1 \vec{b}_1 + \vec{d} = \vec{a}_2 + \lambda_2 \vec{b}_2 \Rightarrow \vec{d} = \vec{a}_2 - \vec{a}_1 + \lambda_2 \vec{b}_2 - \lambda_1 \vec{b}_1 \,.$$

   Taking the cross product of the above expression with e.g. $\vec{b}_1$ ($\vec{b}_1$ and $\vec{b}_2$ are parallel, hence their cross product is 0), and then keeping only the magnitudes of the results, gives

$$|\vec{d}||\vec{b}_1| = |(\vec{a}_2 - \vec{a}_1) \times \vec{b}_1| \Rightarrow |\vec{d}| = |(\vec{a}_2 - \vec{a}_1) \times \hat{b}| \,,$$

   where $\hat{b}$ is the unit vector along the direction of the (parallel) lines.

2. The vector $\vec{q} = a\hat{i} + b\hat{j} + c\hat{k}$ is perpendicular to the plane, so the condition for the line to be parallel to the plane is $\vec{q} \cdot \vec{b} = 0$.

   We need to find the position vector $\vec{p}$ of a point P on the plane (any will do), a point on the line, e.g. A (with position vector $\vec{a}$) and then take the projection of the AP line segment onto the $\vec{q}$ direction. That will give us the distance, $\mathcal{L}$, between the line and the plance.

   One point on the plane is, for example, $(\frac{d}{a}, 0, 0)$, since it satisfies the equation of the plane (assuming ofcourse that $a \neq 0$). Hence the distance is

$$\mathcal{L} = (\vec{p} - \vec{a}) \cdot \frac{\vec{q}}{|\vec{q}|} \,.$$

3. The plane we are looking for is perpendicular to $\vec{AB} = \vec{b} - \vec{a}$ and contains the mid-point M of the line segment AB. We saw in the lectures that the point which divides AB in the ratio $\mu : \lambda$ has position vector

$$\frac{\mu}{\mu + \lambda}\vec{a} + \frac{\lambda}{\mu + \lambda}\vec{b} \,,$$

   so the position vector of M is: $\vec{OM} = (\vec{a} + \vec{b})/2$. Hence, the vector equation of the plane in question is

$$(\vec{r} - \vec{OM}) \cdot \vec{AB} = 0 \Rightarrow \vec{r} \cdot (\vec{b} - \vec{a}) = \frac{1}{2}(\vec{b} + \vec{a})(\vec{b} - \vec{a})$$

$$\Rightarrow \vec{r} \cdot (\vec{b} - \vec{a}) = \frac{1}{2}(|\vec{b}|^2 - |\vec{a}|^2) \,.$$

4. Since

$$\overrightarrow{v} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}\,,$$

we need to express r and θ as a function of time and determine dr/dt and $d\theta/dt$. We have

$$r = \sqrt{x^2 + y^2} = \sqrt{u^2t^2 + 4}$$

and

$$\theta = \arctan\frac{y}{x} = \arctan\frac{2}{ut} \Rightarrow \tan\theta = \frac{2}{ut} \Rightarrow t = \frac{2}{u\tan\theta}\,.$$

Hence

$$\frac{dr}{dt} = \frac{u^2t}{\sqrt{u^2t^2+4}}$$

$$\frac{d\theta}{dt} = \frac{1}{\frac{dt}{d\theta}} = \frac{1}{-\frac{2}{u}\frac{1}{\tan^2\theta}\frac{1}{\cos^2\theta}} = ... = -\frac{2u}{u^2t^2+4}$$

and substituting everything to the first expression for $\overrightarrow{v}$ we get

$$\overrightarrow{v} = \frac{u}{\sqrt{u^2t^2+4}}(ut\hat{r} - 2\hat{\theta})\,.$$

5. (a) In polar coordinates

$$\overrightarrow{v} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}\,,$$

and in this problem $dr/dt = u$, $r = ut$ and $d\theta/dt = \omega$. Hence $\overrightarrow{v} = u\hat{r} + ut\omega\hat{\theta}$.
(b) To express $\overrightarrow{v}$ in cartesian coordinates we need to substitute $\hat{r}$ and $\hat{\theta}$ with $\hat{i}$ and $\hat{j}$. Using also $\theta = \omega t$, we have

$$\hat{r} = \cos\omega t\hat{i} + \sin\omega t\hat{j} \qquad \hat{\theta} = -\sin\omega t\hat{i} + \cos\omega t\hat{j}\,.$$

Hence

$$\overrightarrow{v} = u(\cos\omega t\hat{i} + \sin\omega t\hat{j}) + u\omega t(-\sin\omega t\hat{i} + \cos\omega t\hat{j})$$

$$\Rightarrow \overrightarrow{v} = u(\cos\omega t - \omega t\sin\omega t)\hat{i} + u(\sin\omega t + \omega t\cos\omega t)\hat{j}\,.$$