PHAS1245: Problem Sheet 3 - Solutions

1. One way is to write $\cos^2 x = (1+\cos 2x)/2$ and then

$$\int \cos^2 x\, dx = \frac{1}{2}\int (1+\cos 2x)\, dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) .$$

The other is to use integration by parts:

$$I = \int \cos^2 x\, dx = \int \cos x\, d(\sin x) = \cos x \sin x - \int \sin x\, d(\cos x)$$

$$= \frac{1}{2}\sin 2x + \int \sin^2 x\, dx = \frac{1}{2}\sin 2x + \int (1-\cos^2 x)\, dx = \frac{1}{2}\sin 2x + x - I$$

$$\Rightarrow 2I = \frac{1}{2}\sin 2x + x \Rightarrow I = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) .$$

2. By applying twice the double angle formula we get:

$$\cos^4 x = (\cos^2 x)^2 = \frac{1}{4}(1+\cos 2x)^2 = \frac{1}{4}\left[1 + 2\cos 2x + \frac{1}{2}(1 + cos 4x)\right] .$$

Hence

$$\int \cos^4 x\, dx = \int \left(\frac{3}{8} + \frac{\cos 2x}{2} + \frac{\cos 4x}{8}\right) = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} .$$

3. We saw in the lectures that we can write

$$\frac{x^3}{(x+1)(x-3)} = (x+2) + \frac{1}{4(x+1)} + \frac{27}{4(x-3)} .$$

Hence

$$I = \int \left((x+2) + \frac{1}{4(x+1)} + \frac{27}{4(x-3)}\right) dx = \frac{x^2}{2} + 2x + \frac{1}{4}\ln(x+1) + \frac{27}{4}\ln(x-3)$$

4. Integration by parts:

$$\int \ln x\, dx = x\ln x - \int x\, d(\ln x) = x\ln x - x = x(\ln x - 1)$$

For the second integral, again integrate by parts:

$$\int (\ln x)^2\, dx = x(\ln x)^2 - \int x(2\ln x)\frac{1}{x}\, dx = x(\ln x)^2 - 2\int \ln x\, dx$$

and using the previous result

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2x(\ln x - 1) .$$

5. Substituting $u = -ax^2 \Rightarrow du = -2ax\, dx$, we have

$$I = -\frac{1}{2a}\int \mathrm{e}^u\, du = -\frac{1}{2a}\mathrm{e}^{-ax^2}$$

6. Starting from the previous integral and doing integration by parts

$$\int x\,\mathrm{e}^{-ax^2}\,dx = \int \mathrm{e}^{-ax^2}\,d(\frac{x^2}{2}) = \frac{x^2}{2}\mathrm{e}^{-ax^2} - \int \frac{x^2}{2}\,d(\mathrm{e}^{-ax^2})$$

Now $d(\mathrm{e}^{-ax^2}) = -2ax\mathrm{e}^{-ax^2}\,dx$, and, using also the result from the previous exercise, we get

$$-\frac{1}{2a}\mathrm{e}^{-ax^2} = \frac{x^2}{2}\mathrm{e}^{-ax^2} + \int x^3 a\mathrm{e}^{-ax^2}\,dx$$

$$\Rightarrow a\int_0^\infty x^3\,\mathrm{e}^{-ax^2}\,dx = \left[-\frac{1}{2a}\mathrm{e}^{-ax^2} - \frac{x^2}{2}\mathrm{e}^{-ax^2}\right]_0^\infty \Rightarrow \int_0^\infty x^3\,\mathrm{e}^{-ax^2}\,dx = \frac{1}{2a^2}\,.$$