PHAS1245: Mathematical Methods I - Problem Sheet 7

(Solutions to be handed in at the lecture on Tuesday 27th November 2007)

Staple your answer sheets together and put **your name** and your **tutor's name** on your script (or Dr. Konstantinidis, if you have no tutor in the P&A department).

1. A possible equation of state of a gas takes the form

$$pV = RT\mathrm{e}^{-a/(VRT)}\,,$$

where R and a are constants. Calculate expressions for

$$\left(\frac{\partial p}{\partial V}\right)_T\,, \qquad \left(\frac{\partial V}{\partial T}\right)_p\,, \qquad \left(\frac{\partial T}{\partial p}\right)_V\,,$$

and show that their product is -1. (This is a more general result – see, for example, section 5.4 in the textbook Riley et al).

[Hint: you may need to use implicit differentiation. Also, you may find it easier to take the logarithm of the above expression first.]

2. If $f(x,y) = x^2y^2$ and (ρ,ϕ) the polar coordinates ($x = \rho\cos\phi$, $y = \rho\sin\phi$), show that

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_y + \left(\frac{\partial^2 f}{\partial y^2}\right)_x = \left(\frac{\partial^2 f}{\partial \rho^2}\right)_\phi + \frac{1}{\rho}\left(\frac{\partial f}{\partial \rho}\right)_\phi + \frac{1}{\rho^2}\left(\frac{\partial^2 f}{\partial \phi^2}\right)_\rho\,.$$

(This result is independent of the functional form of f. The quantity $(\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2)$ is called the Laplacian of $f(x,y,z)$ and occurs often in many areas of physics, e.g. electromagnetism and quantum mechanics.)

3. Show that

$$f(x,y) = x^3 - 12xy + 48x + by^2\,, \qquad\qquad b \neq 0\,,$$

has two, one or zero stationary points depending on whether $|b|$ is smaller than, equal to or greater than 3. In the case of two stationary points, find whether these are minima, maxima, or saddle points (the answer depends on the value of b).

4. Find the volume of the largest rectangular parallelipiped (that is a box with edges parallel to the coordinate axes), that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,.$$