PHAS1245: Mathematical Methods I - Problem Class 4

Week starting Monday 19th November

1. (a) Find the angle between the vectors $\vec{A} = (2, 3, -1)$ and $\vec{B} = (2, -1, 2)$.

   (b) Construct unit vectors parallel to $\vec{A}$ and $\vec{B}$ of the previous example.

   (c) Calculate the projection P (or component) of $\vec{A}$ on to (along) $\vec{B}$. What is the projection Q of $\vec{B}$ on to $\vec{A}$?

   (d) Find $\vec{A} \times \vec{B}$ when $\vec{A} = (2, 3, -1)$ and $\vec{B} = (-1, 3, 3)$.

   (e) Hence find the angle between $\vec{A}$ and $\vec{B}$ in the previous example and check your result using the scalar product.

2. Prove Lagrange's identity

$$(\vec{A} \times \vec{B}).(\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})\,.$$

   (Hint: the triple **scalar** product remains unchanged under cyclic permutation of the vectors, since it represents the volume of the parallelepiped defined by the three vectors.)

3. The position of a particle is given by

$$\vec{r} = A(e^{\alpha t}\hat{i} + e^{-\alpha t}\hat{j})\,,$$

   where A and α are constants. Find the magnitude of the velocity at time t.

4. Starting from the expression of velocity in polar coordinates derived in the lectures:

$$\vec{v} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}\,,$$

   derive the expression for the acceleration in polar coordinates.

5. The position vector of a particle is given in polar coordinates by $r = 1/\cos t$, $\theta = t$. Sketch the path of the particle for $0 \le t < \pi/2$ and find its radial and transverse components of accelleration.

6. Particles P_1 and P_2 move around concentric circles of radii a_1 and a_2 in the same sense and with angular velocities ω_1 and ω_2, all respectively. Show that the angular velocity of P_2 about P_1 is given by

$$\Omega = \frac{1}{2}(\omega_1 + \omega_2) + \frac{1}{2}(\omega_1 - \omega_2)\frac{a_1^2 - a_2^2}{r^2}\,,$$

   where $r = P_1P_2$.

   If P_1,P_2 represent two planets, it may be shown that $\omega_1 = \mu^{\frac{1}{2}}/a_1^{\frac{3}{2}}$, $\omega_2 = \mu^{\frac{1}{2}}/a_2^{\frac{3}{2}}$, where μ is a constant for the solar system. Derive that, in this case, the motion of P_2, as observed from P_1, reverses its direction when the angle θ between the radii to the planets is given by

$$\cos\theta = \frac{\sqrt{a_1 a_2}}{a_1 + a_2 - \sqrt{a_1 a_2}}\,.$$