PHAS1245 - Problem Class 3 - Solutions

1. (a) We have

$$|\vec{A}|^2 = A_x^2 + A_y^2 + A_z^2 = 2^2 + 3^2 + (-1)^2 = 14\,,$$
$$|\vec{B}|^2 = B_x^2 + B_y^2 + B_z^2 = 2^2 + (-1)^2 + 2^2 = 9\,,$$
$$\overrightarrow{A}\cdot\overrightarrow{B} = A_xB_x + A_yB_y + A_zB_z = 2\times 2 + 3\times(-1) + (-1)\times 2 = 4 - 3 - 2 = -1\,.$$

But $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$ and substituting in the above numerical values we get

$$-1 = \sqrt{14}\times\sqrt{9}\cos\theta \Rightarrow \cos\theta = \frac{-1}{3\sqrt{14}} \Rightarrow \theta \approx -95^\circ\,.$$

(b) The two unit vectors are

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = (2\hat{\imath} + 3\hat{\jmath} - \hat{k})/\sqrt{14} \qquad \hat{B} = \frac{\vec{B}}{|\vec{B}|} = (2\hat{\imath} - \hat{\jmath} + 2\hat{k})/3\,.$$

(c) Form two right-angle triangles projecting one vector onto the line of the other. Then

$$P = |\vec{A}|\cos\theta = \vec{A}\cdot\vec{B}/|\vec{B}| = -1/3,$$

which is negative since $\theta > \pi/2$. Note that this is not the same as the projection Q of $\overrightarrow{B}$ onto $\overrightarrow{A}$:

$$Q = |\vec{B}|\cos\theta = \vec{A}\cdot\vec{B}/|\vec{A}| = -1/\sqrt{14}.$$

(d)

$$\overrightarrow{A}\times\overrightarrow{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ -1 & 3 & 3 \end{vmatrix} = (9+3)\hat{\imath} - (6-1)\hat{\jmath} + (6+3)\hat{k} = 12\hat{\imath} - 5\hat{\jmath} + 9\hat{k}$$

(e)

$$\left|\overrightarrow{A}\times\overrightarrow{B}\right|^2 = 12^2 + (-5)^2 + 9^2 = 250$$

On the other hand

$$A^2 = 2^2 + 3^2 + (-1)^2 = 14, \quad B^2 = (-1)^2 + 3^2 + 3^2 = 19$$

Hence

$$\sin^2\theta = \frac{250}{14\times 19} \approx 0.9398, \quad \Rightarrow \quad \theta \approx 75.80^\circ$$

Alternatively,

$$\overrightarrow{A}\cdot\overrightarrow{B} = -2 + 9 - 3 = 4$$

and

$$\cos\theta = \frac{4}{\sqrt{14\times 19}}\,,$$

which leads to the same value of the angle θ. The two results are consistent because

$$\sin^2\theta + \cos^2\theta = \frac{250}{266} + \frac{16}{266} = 1\,.$$

2.

$$(\overrightarrow{A} \times \overrightarrow{B}) \cdot (\overrightarrow{C} \times \overrightarrow{D}) = (\overrightarrow{A} \times \overrightarrow{B}) \cdot \overrightarrow{X} = \overrightarrow{X} \times \overrightarrow{A} \cdot \overrightarrow{B} = ((\overrightarrow{C} \times \overrightarrow{D}) \times \overrightarrow{A}) \cdot \overrightarrow{B}$$

$$= (\overrightarrow{D}(\overrightarrow{C} \cdot \overrightarrow{A}) - \overrightarrow{C}(\overrightarrow{D} \cdot \overrightarrow{A})) \cdot \overrightarrow{B} = (\overrightarrow{D} \cdot \overrightarrow{B})(\overrightarrow{C} \cdot \overrightarrow{A}) - (\overrightarrow{C} \cdot \overrightarrow{B})(\overrightarrow{D} \cdot \overrightarrow{A}) .$$

3. Since

$$\overrightarrow{r} = A(e^{\alpha t}\widehat{\imath} + e^{-\alpha t}\widehat{\jmath}), \frac{d\overrightarrow{r}}{dt} = A\alpha(e^{\alpha t}\widehat{\imath} - e^{-\alpha t}\widehat{\jmath}) \text{ and } v_x = A\alpha e^{\alpha t}, v_y = -A\alpha e^{-\alpha t}$$

$$\Rightarrow v = A\alpha \left(e^{2\alpha t} + e^{-2\alpha t}\right)^{\frac{1}{2}} .$$

4. You get the acceleration by simply taking the time derivative of the velocity expressed in polar coordinates. Remember that

$$\frac{d\hat{r}}{d\theta} = \hat{\theta} \qquad \frac{d\hat{\theta}}{d\theta} = -\hat{r} .$$

Ultimately you should get:

$$\overrightarrow{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\hat{\theta} .$$

5. Remembering that the connection between polar and cartesian coordinats is $x = r\cos\theta$ and $y = r\sin\theta$, we find that $x = (1/\cos t)\cos t = 1$, so the trajectory of the particle is the line $x = 1$, parallel to the y-axis.

We have

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{\cos t}\right) = \frac{\sin t}{\cos^2 t} \qquad \frac{d\theta}{dt} = 1 ,$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{\sin t}{\cos^2 t}\right) = \frac{1}{\cos t} + \frac{2\sin^2 t}{\cos^3 t}$$

and ofcourse $d^2\theta/dt^2 = 0$. Then using the formula for acceleration in polar coordinates we have:

$$\vec{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{r} + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\hat{\theta}$$

$$\Rightarrow \vec{a} = \left(\frac{1}{\cos t} + \frac{2\sin^2 t}{\cos^3 t} - \frac{1}{\cos t}\right)\hat{r} + \left(2\frac{\sin t}{\cos^2 t}\right)\hat{\theta} = 2\frac{\sin t}{\cos^2 t}\left(\tan t\hat{r} + \hat{\theta}\right) .$$

6. Bring the inner particle P_1 to rest by applying the vector $\overrightarrow{a_1\omega_1}$ to both particles in the direction shown on the diagram.

Treat P_1 as the origin of (r, θ)(or r, Ω) coordinates.

$$\text{Since } \overrightarrow{v} = \frac{dr}{dt}\widehat{r} + r\frac{d\theta}{dt}\widehat{\theta}$$

we need only consider velocity components perpendicular to $\overrightarrow{r}$ if we require $\frac{d\theta}{dt}$ or Ω.

$$\text{Thus } \ r\frac{d\theta}{dt} = r\Omega = a_1\omega_1\cos\beta + a_2\omega_2\cos\alpha$$

but from the cosine rule we have

$$a_2^2 = a_1^2 + r^2 - 2a_1 r\cos\beta \text{ and } \mathrm{a}_1^2 = \mathrm{a}_2^2 + \mathrm{r}^2 - 2\mathrm{a}_2\mathrm{r}\cos\alpha \ .$$

$$\text{So } r\Omega = \omega_1 a_1 \frac{(-a_2^2 + a_1^2 + r^2)}{2a_1 r} + \frac{\omega_2 a_2(-a_1^2 + a_2^2 + r^2)}{2a_2 r} \ .$$

$$\text{Hence } \ \Omega = \frac{1}{2}(\omega_1 + \omega_2) + \frac{1}{2}(\omega_1 - \omega_2)\left(\frac{a_1^2 - a_2^2}{r^2}\right) \ .$$

The motion reverses its direction when $\Omega = 0$. Putting $\Omega = 0$ and $r^2 = a_1^2 + a_2^2 - 2a_1a_2\cos\theta$ in the above, we get

$$2a_1a_2\cos\theta(a_1^{\frac{3}{2}} + a_2^{\frac{3}{2}}) = (a_1^{\frac{3}{2}} + a_2^{\frac{3}{2}})(a_1^2 + a_2^2) - (a_1^{\frac{3}{2}} - a_2^{\frac{3}{2}})(a_1^2 - a_2^2)$$

The above yields the condition

$$\cos\theta = \frac{a_1^{\frac{1}{2}}a_2 + a_2^{\frac{1}{2}}a_1}{a_1^{\frac{3}{2}} + a_2^{\frac{3}{2}}} = \frac{\sqrt{a_1a_2}}{a_1 + a_2 - \sqrt{a_1a_2}} \ .$$