PHAS1245 - Problem Class 2 - Solutions

1. (a) Set $t = 3x^2 - 2 \Rightarrow dt = 6x\,dx$, hence

$$\int x(3x^2-2)\,dx = \int t\frac{dt}{6} = \frac{1}{6}\frac{t^2}{2} = \frac{(3x^2-2)^2}{12}\,.$$

(b) Set $t = \ln x \Rightarrow dt = dx/x$, hence

$$\int \frac{1}{x\ln^2 x}\,dx = \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{\ln x}\,.$$

(c) Set $t = \sin\theta \Rightarrow dt = \cos\theta\,d\theta$. Also, **don't forget to change the limits of integration** unless you re-express the result in terms of θ. So

$$\int \cos^3\theta\,d\theta = \int (1-\sin^2\theta)\cos\theta\,d\theta = \int (1-t^2)dt = t - \frac{t^3}{3} = \sin\theta - \frac{\sin^3\theta}{3}$$

and for our definite integral we get

$$I = \left[\sin\theta - \frac{\sin^3\theta}{3}\right]_{-\pi/2}^{\pi/2} = 1 + 1 - \frac{1+1}{3} = \frac{4}{3}\,.$$

(d) Set $t = \cos x \Rightarrow dt = -\sin x\,dx$, hence

$$\int \mathrm{e}^{\cos x}\sin x\,dx = -\int \mathrm{e}^t\,dt = -\mathrm{e}^{\cos x}\,.$$

(e) set $u = 1/t \Rightarrow du = -dt/t^2$. Hence

$$\int \frac{1}{t^2}\sin\left(\frac{1}{t}\right)\,dt = -\int \sin u\,du = \cos u = \cos\left(\frac{1}{t}\right)\,.$$

(f) Write $\sin 2x = 2\sin x\cos x$ and then set $t = \cos x \Rightarrow dt = -sinx\,dx$. Like this we get

$$\int \frac{\sin 2x}{1+\cos^2 x}\,dx = -\int \frac{2t\,dt}{1+t^2} = -\int \frac{d(t^2+1)}{t^2+1} = -\ln(t^2+1) = -\ln(\cos^2 x+1)\,.$$

2. For

$$I = \int \frac{\ln(a^2 + x^2)}{x^2}\, dx$$

we choose

$$u = \ln(a^2 + x^2) \Rightarrow du = \frac{2x}{a^2 + x^2}\, dx \quad \text{and} \quad dv = \frac{dx}{x^2} \Rightarrow v = -\frac{1}{x}$$

Hence

$$I = -\frac{\ln(a^2 + x^2)}{x} + \int \frac{1}{x} \frac{2x}{a^2 + x^2}\, dx = -\frac{\ln(a^2 + x^2)}{x} + \frac{2}{a} \arctan(\frac{x}{a})\,.$$

The integral

$$I = \int x^3 (1 - x^2)^3\, dx$$

can be done by brute force (expand the cubic etc.), but an alternative is the following: first set $t = x^2 \Rightarrow dt = 2x\, dx$, and the integral becomes $\int t(1-t)^3\, dt$. Then choose: $u = t$ and $dv = (1-t)^3 \Rightarrow v = -(1-t)^4/4$. Thus

$$I = -\frac{t(1-t)^4}{4} + \frac{1}{4} \int (1-t)^4\, dt = -\frac{t(1-t)^4}{4} - \frac{1}{4}\frac{1}{5}(1-t)^5$$

$$\Rightarrow I = -\frac{(1-x^2)^4 (4x^2 + 1)}{20}\,.$$

For

$$I = \int x^r \ln x\, dx\,, \quad r \neq -1$$

choose $u = \ln x \Rightarrow du = dx/x$ and $dv = x^r\, dx \Rightarrow v = x^{r+1}/(r+1)$. Then

$$I = \frac{x^{r+1}}{r+1} \ln x - \int \frac{x^{r+1}}{(r+1)} \frac{1}{x}\, dx = \frac{x^{r+1}}{r+1} \ln x - \frac{1}{r+1} \int x^r\, dx$$

$$\Rightarrow I = \frac{x^{r+1}}{r+1} \ln x - \frac{1}{(r+1)^2} x^{r+1} = \frac{x^{r+1}}{r+1} \left(\ln x - \frac{1}{r+1} \right)\,.$$

3.

$$I = \int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a}\int \frac{1}{x^2+\frac{b}{a}x+\frac{c}{a}}\,dx = \frac{1}{a}\int \frac{1}{(x+\frac{b}{2a})^2 - \frac{b^2-4ac}{4a^2}}\,dx\,.$$

(i) If $b^2 > 4ac$,

$$I = \frac{1}{a}\int \frac{1}{(x-x_1)(x-x_2)}\,dx$$

where $x_{1,2} = -b \pm \sqrt{b^2-4ac}/(2a)$. We then break the integrand into two fractions:

$$I = \frac{1}{a(x_1-x_2)}\int \left(\frac{1}{x-x_1} - \frac{1}{x-x_2}\right)dx = \frac{1}{a(x_1-x_2)}\left[\ln(x-x1) - \ln(x-x2)\right]$$

(ii) If $b^2 = 4ac$, we have

$$I = \frac{1}{a}\int \frac{1}{(x+\frac{b}{2a})^2}\,dx = -\frac{1}{a}\frac{1}{x+\frac{b}{2a}} = -\frac{2}{2ax+b}\,.$$

(iii) If $b^2 < 4ac$, the integral has the form

$$I = \frac{1}{a}\int \frac{1}{t^2+k^2} = \frac{1}{k}\arctan\left(\frac{t}{k}\right)$$

with $t = x + b/2a$ and $k = \sqrt{4ac-b^2}/2a$, hence

$$I = \frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\,.$$

4.

$$\sinh^{-1} x = u \Rightarrow x = \sinh u = \frac{1}{2}(\mathrm{e}^u - \mathrm{e}^{-u})$$

$$\Rightarrow 2x = \mathrm{e}^u - \mathrm{e}^{-u} \Rightarrow \mathrm{e}^{2u} - 2x\mathrm{e}^u - 1 = 0$$

Now set $t = \mathrm{e}^u$ and solve the quadratic $t^2 - 2xt - 1 = 0$. You will get

$$(t - x + \sqrt{x^2+1})(t - x - \sqrt{x^2+1}) = 0$$

But $t = \mathrm{e}^u > 0$, so only the second factor gives real solutions $t = x + \sqrt{x^2+1}$, and taking the logarithm of this we arrive at the requested expression for $\sinh^{-1} x$.

5. This smells inverse hyperbolic functions (as we have seen in the lectures). So, first bring the integrand of I to the form $\sqrt{x^2 \pm a^2}$:

$$\sqrt{x^2+4x\pm 4+13} = \sqrt{(x+2)^2+3^2}\,.$$

Then, you either look at your notes and see that this is of the form

$$\int \sqrt{t^2+a^2}\,dt = \frac{a^2}{2}\left[\sinh^{-1}(\frac{t}{a}) + \frac{t\sqrt{t^2+a^2}}{a^2}\right]$$

with $t = x + 2$ and $a = 3$, or, better, you solve it again, by making the substitution $(x+2) = 3\sinh t \Rightarrow dx = 3\cosh t\, dt$ and $\sqrt{\sinh^2 t + 1} = \cosh t$. Hence the integral becomes

$$I \int (3\cosh t)(3\cosh t)\, dt = 9\int \cosh^2 t\, dt = 9\int \cosh t\, d(\sinh t)$$

$$\Rightarrow I = 9\left[\cosh t \sinh t - \int \sinh t\, d(\cosh t)\right] = 9\cosh t \sinh t - 9\int \sinh^2 t\, dt\,,$$

and using $\cosh^2 t - \sinh^2 t = 1$ we get

$$I = 9\sinh t\sqrt{1+\sinh^2 t} + 9t - I$$

$$\Rightarrow I = \frac{9}{2}\left[\frac{(x+2)\sqrt{9+(x+2)^2}}{9} + \sinh^{-1}\left(\frac{x+2}{3}\right)\right]$$

6. (a) The shape of the container is $r(h) = 0.39h^{1/4}$ or $h(r) = (r/0.39)^4$, hence it is the volume formed by rotating a quartic function around the y axis. This is a "volume of revolution". The volume of an infinitesimal slice of thickness dh, contained between two horizontal planes at height h_i and $h_i + dh$ is $dV = \pi r^2 dh$, hence the volume of the container is

$$V = \int_0^{0.5} \pi r^2\, dh = \int_0^{0.5} \pi 0.39^2 h^{1/2}\, dh = 0.39^2\pi\left[\frac{h^{3/2}}{3/2}\right]_0^{0.5} \Rightarrow V = 0.11\, m^3\,.$$

(b) For the water level to fall at a uniform rate, we must show that

$$\frac{dh}{dt} = \text{constant}\,.$$

Assuming that in time dt the height changes by dh we have

$$dh = \frac{dh}{dt}\, dt = \frac{dh}{dV}\frac{dV}{dt}\, dt\,.$$

We are given dV/dt and from (a) we have

$$dh/dV = \frac{1}{\frac{dV}{dh}} = \frac{1}{\pi r^2}\,.$$

Hence

$$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt} = \frac{1}{\pi r^2}(-0.003\sqrt{h}) = \frac{1}{0.39^2\pi\sqrt{h}}(-0.003\sqrt{h}) = \frac{-0.003}{0.39^2\pi} = \text{constant}\,,$$

i.e. the water level falls at a uniform rate (hence the name water clock!). To find the total time it takes to empty the container, we need dt/dh, since if the change by dh take time $dt = \frac{dt}{dh}\, dh$, so the total time is

$$T = \int_{0.5}^{0} \frac{dt}{dh}\, dh = \int \frac{-0.003}{0.39^2\pi}\, dt = \frac{-0.003}{0.39^2\pi}(-0.5) = 79.6\,\text{hours}\,.$$

(Note that the integration goes from 0.5 -full container- to 0).