PHAS1245 - Problem Class 2 - Solutions

1. (a) We have

$$y = x^2 + 2 + x^{-2} \quad \text{and} \quad \frac{dy}{dx} = 2x - \frac{2}{x^3} \ .$$

(b) We have

$$y = x^{\frac{1}{2}} + x^{-\frac{1}{2}} \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}}\left(1 - \frac{1}{x}\right) \ .$$

(c) We have

$$y = (x^{\frac{1}{2}}+1)(x^{\frac{1}{2}}-1)^{-1} \quad , \quad \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x^{\frac{1}{2}}-1)^{-1} + (x^{\frac{1}{2}}+1)(-1)(x^{\frac{1}{2}}-1)^{-2}\frac{1}{2}x^{-\frac{1}{2}}$$

$$\text{and} \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}\frac{(\sqrt{x}-1)-(\sqrt{x}+1)}{(\sqrt{x}-1)^2} = \frac{1}{\sqrt{x}}\frac{-1}{(\sqrt{x}-1)^2} \ .$$

(d) First, if $y = sin^{-1}x$ then

$$x = sin\, y \ , \ dx = cos\, y\, dy = (\sqrt{(1-sin^2 y})dy = \sqrt{(1-x^2}\, dy \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}} \ .$$

$$\text{Then} \quad \frac{d}{dx}sin^{-1}x^2 = \frac{1}{\sqrt{(1-x^4)}} \times 2x = \frac{2x}{\sqrt{(1-x^4)}} \ .$$

(e) We have

$$y = ln\, \frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}} \quad \text{and} \quad \frac{dy}{dx} = \frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}+\sqrt{x}}\frac{d}{dx}((\sqrt{a}+\sqrt{x})(\sqrt{a}-\sqrt{x})^{-1})$$

$$= \frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}+\sqrt{x}}\left(\frac{1}{2}x^{-\frac{1}{2}}(\sqrt{a}-\sqrt{x})^{-1} + \frac{(\sqrt{a}+\sqrt{x})(-1)(-\frac{1}{2})x^{-\frac{1}{2}}}{(\sqrt{a}-\sqrt{x})^2}\right)$$

$$= \frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}+\sqrt{x}}\ \frac{1}{2}\ \frac{1}{\sqrt{x}}\frac{\sqrt{a}-\sqrt{x}+\sqrt{a}+\sqrt{x}}{(\sqrt{a}-\sqrt{x})^2} = \frac{1}{\sqrt{x}}\frac{\sqrt{a}}{(a-x)} \ .$$

It is much easier here to start from

$$y = ln\, \frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}} = ln\ (\sqrt{a}+\sqrt{x}) - ln\ (\sqrt{a}-\sqrt{x}) \qquad !!!$$

2. At the points of interseption $f() = g(x)$. Hence

$$x^2 = 1 - x^2 \Rightarrow 2x^2 = 1 \Rightarrow x_{1,2} = \pm\frac{1}{\sqrt{2}}$$

and since $y = x^2$, the two points of interception are:

$$(x_1, y_1) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad \text{and} \quad (x_2, y_2) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Now we need the derivatives of the functions at these two points. These derivatives are the gradients of the lines tangent to the curves at the given points, hence they give the tan of the angle between these lines and the x-axis. **(This is something that I was going to show in an example during the lectures, but I ran out of time so I just mentioned it and asked them to try it at home. You may find that some stundets are not familiar with the connection between the gradient of a line and the tangent of the angle it makes with the x-axis.)**

$$f'(x) = 2x \Rightarrow f'(x_1) = 2x_1 = -2/\sqrt{2} = -\sqrt{2} = \tan\theta_1$$

$$g'(x) = -2x \Rightarrow g'(x_1) = -2x_1 = 2/\sqrt{2} = \sqrt{2} = \tan\theta_2$$

and the angle between these lines is $\theta = \theta_1 - \theta_2$, hence

$$\tan\theta = \tan(\theta_1 - \theta_2) = \frac{tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} \Rightarrow \tan\theta = \frac{-2\sqrt{2}}{1-2} = 2\sqrt{2}\,.$$

The work is the same for the other point, so I don't include it here. If the students do not have time in the problem class they can do just one of the points and leave the other for doing at home.

3. We have

$$x = a(\theta - \sin\theta) \Rightarrow \frac{dx}{d\theta} = a(1 - \cos\theta)$$

$$y = a(1 - \cos\theta) \Rightarrow \frac{dy}{d\theta} = a\sin\theta$$

So,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{1-(1-\sin^2(\theta/2))} = \cot\frac{\theta}{2}$$

To sketch the curve we need a few points:

θ	x	y	tangent
0	0	0	∞
$\pi/2$	$a(\pi/2 - 1)$	a	1
π	$a\pi$	$2a$	0
$3\pi/2$	$a(3\pi/2 + 1)$	a	-1
2π	$2a\pi$	0	∞

4. Easier to use logarithnmic differentiation:

$$\ln y = \ln x + 2x\ln a + x^2 \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + 2\ln a + 2x$$

$$\Rightarrow \frac{dy}{dx} = a^{2x} e^{x^2} + x a^{2x} e^{x^2} 2 \ln a + 2x^2 a^{2x} e^{x^2}$$

and requiring this to be 0, we get:

$$2x^2 + 2x \ln a + 1 = 0\,.$$

The quadratic has $\Delta = 4(\ln a)^2 - 8$ and hence it has two real roots if $\Delta > 0 \Rightarrow |\ln a| > \sqrt{2}$, one when $(\ln a)^2 = 2$ and none otherwise. In the case of a single solution that is $x = -\ln a/2$, so when $\ln a = -\sqrt{2}$ the stationary point is at $x = \sqrt{2}/2$ and for $\ln a = \sqrt{2}$ the stationary point is at $x = -\sqrt{2}/2$.