PHAS1245 - Problem Class 1 - Solutions

1. If x_1 and x_2 are roots then

$$(x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1 x_2 = 0 \ .$$

Comparing this with $ax^2 + bx + c = x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, we find

$$x_1 + x_2 = -\frac{b}{a} \quad \text{and} \quad x_1 x_2 = \frac{c}{a} \ .$$

2. (a) Let

$$\frac{x^2 + 3}{x(x^2 + 2)} = \frac{A}{x} + \frac{Bx + C}{(x^2 + 2)} \ ,$$

thus $x^2 + 3 = A(x^2 + 2) + (Bx + C)x$.

Choosing $x = 0$ we find $A = \frac{3}{2}$ and $x^2 + 3 = \frac{3}{2}x^2 + 3 + Bx^2 + Cx$.

Comparing the coefficients of x^2 , we find $B = -\frac{1}{2}$.

Now choose $x = 0$ again to find that $C = 0$.

$$\text{Hence} \quad \frac{x^2 + 3}{x(x^2 + 2)} = \frac{3}{2x} - \frac{x}{2(x^2 + 1)}$$

(b) Let

$$\frac{3}{x(3x - 1)^2} = \frac{A}{x} + \frac{B}{(3x - 1)} + \frac{C}{(3x - 1)^2}$$

or $3 = A(3x - 1)^2 + Bx(3x - 1) + Cx$.

Choosing $x = 0$ we find $A = 3$.

Choosing $x = \frac{1}{3}$ we find $C = 9$.

Then setting $x = 1$ we fine $B = -9$.

$$\text{Hence} \quad \frac{3}{x(3x - 1)^2} = \frac{3}{x} - \frac{9}{(3x - 1)} + \frac{9}{(3x - 1)^2}$$

3. (a) Since $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$$

(since $\cos^2 A = 1 - \sin^2 A$) .

Since $\sin(A + B) = \sin A \cos B + \cos A \sin B$,

$$\sin 2A = 2\sin A\cos A \quad \text{and} \quad \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$= 2\tan\frac{A}{2}\cos^2\frac{A}{2} = \frac{2\tan\frac{A}{2}}{\sec^2\frac{A}{2}} = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}} \ .$$

$$\text{Since} \quad \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = \cos^2\frac{A}{2}(1-\tan^2\frac{A}{2}) = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}} \ .$$

(b) For $\cos 2\theta + 3\sin\theta = 2$ use $\cos 2\theta = 1 - 2\sin^2\theta$ to obtain $1 - 2\sin^2\theta + 3\sin\theta$.

$$\text{i.e.} \ \ 2\sin^2\theta - 3\sin\theta + 1 = (2\sin\theta - 1)(\sin\theta - 1) = 0$$

$$\text{and} \quad \sin\theta = \frac{1}{2} \ \text{ or } \ 1 \ \text{ i.e. } \ \theta = \frac{\pi}{6}, \frac{5\pi}{6} \ \text{ or } \ \frac{\pi}{2} \ .$$

For $\sin\theta + 2\cos\theta = 1$ use the expression for $\sin\theta$ and $\cos\theta$ in terms of $\tan\frac{\theta}{2}$. Let $\tan\frac{\theta}{2} = t$. Then

$$\frac{2t}{1+t^2} + \frac{2(1-t^2)}{(1+t^2)} = 1$$

which yields $3t^2 - 2t - 1 = (3t+1)(t-1) = 0$.

$$\text{So} \quad \tan\frac{\theta}{2} = -\frac{1}{3} \ \text{ or } \ 1 \ \text{ and } \ \frac{\theta}{2} = 161.57^\circ \ \text{ or } \ \frac{\theta}{2} = 45^\circ \ ,$$

$$\text{i.e.} \quad \theta = 323.14^\circ \ \text{ or } \ 90^\circ \ .$$

(Note - substituting for *sine* in the above would involve a square root and a loss of sign information on squaring.)

(c)

$$\sin\theta + \sin 4\theta = 2\sin\frac{\theta+4\theta}{2}\cos\frac{\theta-4\theta}{2} = 2\sin\frac{5\theta}{2}\cos\frac{3\theta}{2}$$

and

$$\sin 2\theta + \sin 3\theta = 2\sin\frac{2\theta+3\theta}{2}\cos\frac{2\theta-3\theta}{2} = 2\sin\frac{5\theta}{2}\cos\frac{\theta}{2} \ .$$

Hence we get

$$\sin\frac{5\theta}{2}\left(\cos\frac{3\theta}{2} - \cos\theta 2\right) = 0 \Rightarrow \sin\frac{5\theta}{2}\left(-2\sin\theta\sin\frac{\theta}{2}\right) = 0$$

So, $\theta = 0$ is a triple solution and the others are:

$$\frac{5\theta}{2} = n\pi \Rightarrow \theta = 5n\pi/2$$

there are no values of n other than 0 that are within the $(-\pi, \pi]$.

$$\theta = n\pi \Rightarrow n = 0, 1$$

and $\theta = pi$ is another solution, and finally

$$\theta/2 = n\pi$$

which again is only satisfied for $n = 0$ in the required range.

4. We have

$$\epsilon_1 = 1 - \frac{T_{C1}}{T_H} \quad \text{and} \quad \epsilon_2 = 1 - \frac{T_{C2}}{T_H}$$

$$\Delta\epsilon = \epsilon_2 - \epsilon_1 = \frac{T_{C1} - T_{C2}}{T_H}$$

which is negative if $T_{C2} > T_{C1}$

$$\frac{\Delta\epsilon}{\epsilon_1} = \frac{T_{C1} - T_{C2}}{T_H} \frac{T_H}{T_H - T_{C1}} = \frac{T_{C1} - T_{C2}}{T_H - T_{C1}}$$

For a gas power plant $\frac{\Delta\epsilon}{\epsilon_1} = \frac{-15}{385} = 0.038$ i.e. 3.8%.

For a PWR nuclear plant $\frac{\Delta\epsilon}{\epsilon_1} = \frac{-15}{235} = 0.064$ i.e. 6.4%.

5. The formula for the binomial coefficients, which the students should look up in their notes,is

$$^nC_k = \frac{n!}{k!(n-k)!} \, .$$

Hence

$$^6C_0 = \frac{6!}{0!6!} = 1$$

$$^6C_1 = \frac{6!}{1!(5)!} = 6 =^6 C_5$$

$$^6C_2 = \frac{6!}{2!(4)!} = \frac{5 \cdot 6}{2} = 15 =^6 C_4$$

$$^6C_3 = \frac{6!}{3!(3)!} = \frac{4 \cdot 5 \cdot 6}{2 \cdot 3} = 20$$

Hence

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \, .$$

6. Sound takes time to travel from the car to the observer, Thus if the observer receives the pulse at time t, then the (retarded) time this signal is emitted is t minus the time the sound takes to cover the distance between the retarded position of the car to the observer.

$$\text{Thus} \quad [t] = t - \frac{[|\overrightarrow{r}|]}{c_s} \quad \text{where} \quad [|\overrightarrow{r}|] = \sqrt{(z - v[t])^2 + x^2 + y^2}$$

(where $z - v[t]$ is the retarded z position of car)

$$\text{i.e.} \quad c[t] - ct = [\overrightarrow{r}]$$

and squaring

$$c^2[t] - 2ct[t] + c^2t^2 = z^2 + v[t]^2 - 2zv[t] + x^2 + y^2$$

or

$$(c^2 - v^2)[t]^2 + 2(zv - c^2t)[t] - (x^2 + y^{2)} - z^2 + c^2t^2 = 0$$

and

$$[t]^2 + \frac{2(zv - c^2t)}{(c^2 - v^2)}[t] - \frac{(x^2 + y^2) + z^2 - c^2t^2}{(c^2 - v^2)} = 0$$

The above is a quadratic in $[t]$, which we now solve by completing the square. We find

$$\left[[t] + \frac{(vz - c^2t)}{c^2 - v^2}\right]^2 = \frac{(vz - c^2t)^2 + (c^2 - v^2)(z^2 + (x^2 + y^2) - c^2t^2)}{(c^2 - v^2)^2}$$

The above, taking the negative square to ensure retardation and a very steady nerve yields

$$[t] = \frac{t - \frac{vz}{c^2} - \frac{1}{c}\sqrt{(z - vt)^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)}}{(1 - \frac{v^2}{c^2})}$$