Space Time and Gravity (STG — PHY308)

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A WARNING, these notes may be dangerous to your intellectual health

These notes are a very rough aid memoire to the course. They are no substitute to attending the lectures and making your own notes. Relying on these notes alone will lead to failure. You should do the exercises, be active in class so that you ask questions as they arise. This subject perhaps more than most builds upwards and so you must understand each step. As with all notes, they are not a complete record of the lectures and there may be erros of various kinds. No notes are error free..The internet is not error free..Books are not error free. Think and understand and then your will spot errors as opposed to propagating them. This is one of the points of education; do not rely on the authority of the text or god-forbid wikipedia. That said if you do spot an error in these notes email me and they will be fixed (I hope).

A big thank you to Alexander Hamilton who did the great majority of Tex writing.

Goals of the course

- Be able to do calculations in General Relativity

- Consequences: Black holes, Cosmology etc.
- Conceptual understanding of key ideas

Plan

- Introduction of physical concepts & data, Principle of Equivalence
- Special Relativity reworked.
- Curved spacetime
- Tensors
- Covariant Derivative
- Geodesic Equation
- Riemann Curvature
- Einstein Field Equations
  - Dust
  - Fluid
  - Schwarzschild $\Rightarrow$ Black Holes
- Cosmology Tests:
  - Perihelion Shift of Mercury
  - Bending Light
  - Gravitational Lensing
  - Waves, Time Dilation

Books

- James B Hartle "Gravity, An introduction to Einstein's General Relativity"

Need for GR

- S.R only valid for inertial frames.
- How does relativity generalise when we use non-inertial frames ie accelerations?
- We will see connection between accelerated frames and gravity.

More Properties

We can introduce a potential $\phi(x)$ per unit mass.

$$F := -\vec{\nabla}\phi \text{ per unit mass}$$

Obeys

$$\begin{aligned}
\nabla^2\phi &= 4\pi G\rho \qquad \text{Gravitational version of Poisson's Equ.} \\
\int_V \nabla^2\phi dV &= \int_V 4\pi G\rho \\
\int_V \left(\vec{\nabla}\phi\right)\cdot d\vec{A} &= 4\pi M \\
\frac{GM}{\cancel{r^2}}4\pi\cancel{r^2} &= 4\pi M
\end{aligned}$$

Newton

$$|F_{\text{grav}}| = \frac{G_N m_1 m_2}{T_{12}^2}$$

$$G_N = 6.67 \times 10^{-11}\ \text{m}^3\ \text{kg}^{-1}\ \text{s}^{-1}$$

Some properties, gravity is unscreened ie. there are no "negative charges".

Weak, E.G. Two Protons:

$$\frac{F_{\text{grav}}}{F_{\text{elec}}} = \frac{G_N m_p^2/r^2}{e^2/4\pi\epsilon_0 r^2} \sim 10^{-36}$$

$$\text{c.f. } \frac{F_{\text{mag}}}{F_{\text{elec}}} \sim \frac{v^2}{\mu_0\epsilon_0} \sim \frac{v^2}{c^2}$$

Newtonian Law gives action at a distance.

Not allowed by relativity.

In EMF this was solved by introducing a field & as a result there was radiation EM waves.

In G.R. there is Geometry but again there will be radiation, gravitational waves.

We will see later that G.R. will become important for objects where $\frac{GM}{Rc^2}$ is not too small. c.f. $\left(\frac{v}{c}\right)$

$$\left.\begin{array}{lll} \text{Earth} & \frac{GM_{\text{E}}}{R_E^2} & \sim\ 10^{-9} \\ \text{Sun} & \frac{GM_{\text{Sun}}}{R_{\text{Sun}}^2} & \sim\ 10^{-6} \end{array}\right\} \quad \text{Small but detectable changes to Newtonian gravity}$$

$$\left.\begin{array}{ll} \text{Neutron Star} & \sim\ 0.1 \\ \text{Black Holes} & \sim\ 1 \end{array}\right\} \quad \text{Truely relativistic}$$

Early Universe $\sim$ Large

A remark: Quantum Gravity

At scales $l_p = (G\hbar/c^3)^{1/2} = 1.62 \times 10^{-35}$ m

Then G.R will breakdown & we need Quantum Theory.
This is just as for other areas of physics where the very small requires quantum theory.
Quantum Gravity is problematic; only known cure is string theory.

	$L \gg 1$	$L \ll 1$
$M \ll 1$	Newton Classical	Quantum
$M \gg 1$	G.R. Classical	Quantum Gravity

Basic Concepts in G.R.

The Principle of Equivalence

"At every point in spacetime, in an arbitrary gravitational field, it is possible to choose a locally inertial coordinate system (a freely falling frame) such that within a sufficiently small region around that point the laws of nature take the same form as in the unaccelerated coordinate system in the absence of gravity."

Note, only a local freely falling frame, a global choice is not always possible.

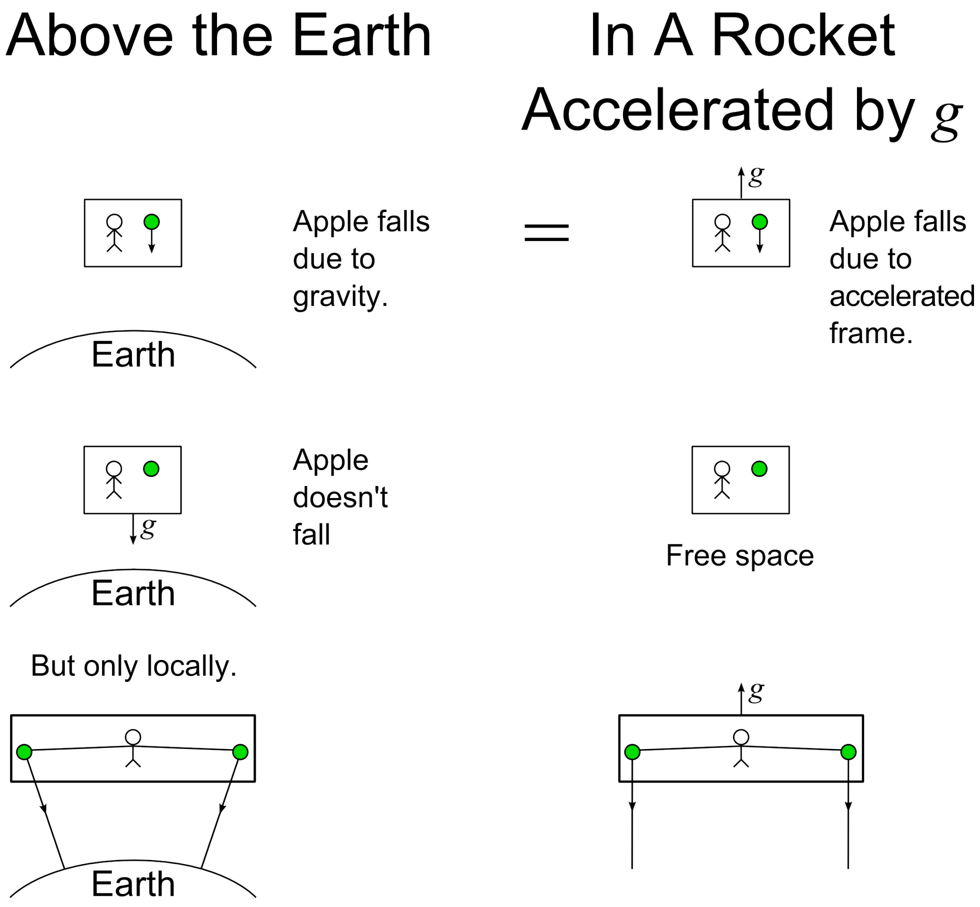

Figure 1:

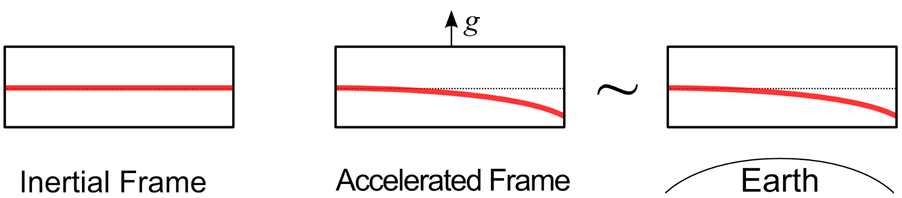

Figure 2: When watching a light beam travel, an accelerated frame is equivilent to being in a gravitational field.

Mathematical descriptions of non-local effect;

"Newtonian tidal equation".

Write using gravitational potential unit masses

$$\ddot{x} = -\vec{\nabla}\phi\left(x\right) \quad \text{c.f. } \ddot{x} = -\vec{\nabla}\phi\left(x\right) \text{for electric potential.}$$

Consider two particles ζ apart

$$\vec{y} = \vec{x} + \vec{\zeta}$$

So

$$\ddot{\vec{y}} = \ddot{\vec{x}} + \ddot{\vec{\zeta}}$$

Using above:

$$\begin{array}{rcccl} \ddot{\vec{y}} & = & -\vec{\nabla}\phi\left(\vec{y}\right) & & \\ \therefore \quad \ddot{y}^i & = & -\frac{\partial}{\partial y^i}\phi\left(y\right) & = & -\frac{\partial}{\partial x^i}T\phi\left(x+y\right) \end{array}$$

Now Taylor Expand

$$= \underbrace{-\frac{\partial}{\partial x^i}\phi\left(x\right)}_{=\ddot{x}^i} - \sum_j \zeta^i \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\phi\left(x\right)$$

$$\therefore \cancel{\ddot{x}^i} + \ddot{\zeta}^i = \cancel{\ddot{x}^i} - \sum_j \zeta^j K_{ij}$$

$$K_{ij} \equiv +\frac{\partial}{\partial x^i \partial x^j}\phi\left(x\right)$$

$$\therefore \ddot{\zeta}^i \equiv K_{ij}\zeta^j$$

N.B.

$\text{Tr}\left[K\right] = \nabla^2\phi = 4\pi G_N \rho_M$ c.f. EMF $\nabla^2\phi = \rho_Q/\epsilon_0$

Review the Logic:

1. S.R. is for inertial frames.
2. Want to consider non-inertial frames (hence G.R.)

3. Principle of equivalence shows link to gravity
4. Also the importance of a non-globally defined freely falling frame in a gravitational field.

Back to Special Relativity

Lorentz Transformations ($c = 1$)

$$x' = \frac{x - vt}{\sqrt{1 - v^2}}$$
$$t' = \frac{t - vx}{\sqrt{1 - v^2}}$$

Note about $c = 1$ units. This is the most natural unit since as we will see x & t "rotate" into eatch other simply in those units.

Rotating Reference Frames and invariance's.

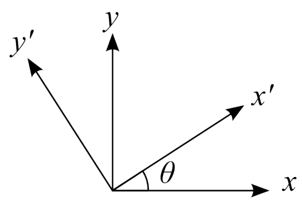

Figure 3:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix}$$

x', y' are obviously different in the rotated coordinates.

But what will be the same is the overall distance

$$s^2 = x^2 + y^2 = x'^2 + y'^2$$

This is an “invariant” of rotations.

In 3-d $\qquad ds^2 = dx^2 + dy^2 + dz^2$

Now lets consider what happens if we change a sign in one of the coordinates

$$ds^2 = -dx^2 + dy^2 + dz^2$$

What transformations leave ds’ invariant?

$$\begin{array}{rcl} t' & = & t\cosh\theta - x\sinh\theta \\ x' & = & -t\sinh\theta - x\cosh\theta \end{array} \qquad\qquad \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}$$

then

$$\begin{array}{rcl} -t'^2 + x'^2 & = & -t^2 + x^2 \\ & = & -t^2\cosh^2\theta - x^2\sinh^2\theta - \cancel{2xt\cosh\theta\sinh\theta} \\ & & +\cancel{2xt\cosh\theta\sinh\theta} + t^2\sinh^2\theta + x^2\cosh^2\theta \\ & = & -t^2\left(\cosh^2\theta - \sinh^2\theta\right) + x^2\left(\cosh^2\theta - \sinh^2\theta\right) \\ & = & -t^2 + x^2 \end{array}$$

$$\cosh^2\theta - \sinh^2\theta \equiv 1$$

Now put into our Lorentz Const to see to see this is the same.

$$\frac{1}{\sqrt{1-\tanh^2\theta}} = \frac{1}{\sqrt{\operatorname{sech}^2\theta}} = \frac{1}{\operatorname{sech}\theta} = \cosh\theta$$
$$\cosh\theta\tanh\theta \equiv \sinh\theta$$

So

$$x' = \frac{x - vt}{\sqrt{1-v^2}} \qquad \& \qquad t' = \frac{t - vx}{\sqrt{1-v^2}}$$

So the velocity is $\tanh\theta$ where θ is the “rotation” angle between space and time.

This is equivalent to

$$\begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus in S.R. $ds^2 = -dt^2 + dx^2 + dy^2 + dx^2$

in the invariant which gives distances in spacetime.

This is Minkowski space.

Our usual $v = 0$ frame is just like picking a direction, it is nothing special, physics cannot depend on it.

Notation

$$x^\mu = \begin{pmatrix} t \\ x^i \end{pmatrix} \qquad \text{four vector}$$

Make a scalar product using:

$$\sum_{\mu,\nu} x^\mu \eta_{\mu\nu} x^\nu \qquad \text{where} \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note, scalar product is the same in all frames.

$\therefore ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu$

$\eta_{\mu\nu}$ is called the metric (of Minkowski space)

$\eta^{\mu\nu}$ is $(\eta)^{-1}$ the inverse metric.

Einstein convention: Repeated indices are summed over.

So

$$x^\mu \eta_{\mu\nu} = \sum_\mu x^\mu \eta_{\mu\nu} \text{ etc.}$$

Note

$$\begin{array}{lcll} x \cdot x & > & 0 & \text{Spacelike} \\ x \cdot x & = & 0 & \text{Null or Lightlike}, \quad \text{note } x \neq 0 \\ x \cdot x & < & 0 & \text{Timelike} \end{array}$$

Changing Coords

$$\tilde{x}^{\mu}(x)$$

that is the new coodinates $\tilde{x}^{\mu}$ are a function of the old coordinates x^{ν}.
Then calculate derivatives (one forms)

$$dx = dr\cos\theta - r\sin\theta \;\; d\theta$$
$$dy = dr\sin\theta - r\cos\theta \;\; d\theta$$

Put into

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= dr^2 + r^2 \;\; d\theta^2 \end{aligned}$$

After using

$$\cos^2\theta + \sin^2\theta \equiv 1$$

Extract new metric:

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

A vector is with indices up

$$\text{eg } x^\mu = \begin{pmatrix} t \\ x^i \end{pmatrix}$$

The transpose is with indices down e.g. $x_\mu = (-t, x_i)$

A matrix has both up & down e.g. $M^\nu{}_\mu$

The metric allows us to raise & lower indices.

So, $x_\mu = \eta_{\mu\nu} x^\nu, x^\mu = \eta^{\mu\nu} x_\nu$

To add/sum over indices have one up & one down

e.g. $x_\mu x^\mu$, $x^\nu = M^\nu{}_\mu x^\mu$ etc.

What is the metric?

The metric is how you measure distance for a set of coordinates, (it depends on coordinate choice).

e.g. Cartesian:

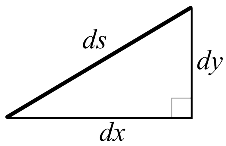

Figure 4:

$$ds^2 = dx^2 + dy^2$$

Polar:

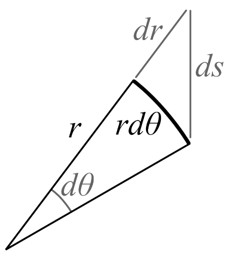

Figure 5:

$$ds^2 = dr^2 + r^2 d\theta^2$$

Metric here would be $\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

Now we see the possibility of the metric being a function of where you are in spacetime.

This in general means we should try to understand curved geometry. To see how this relates to gravity consider the following:

The surface of a ball

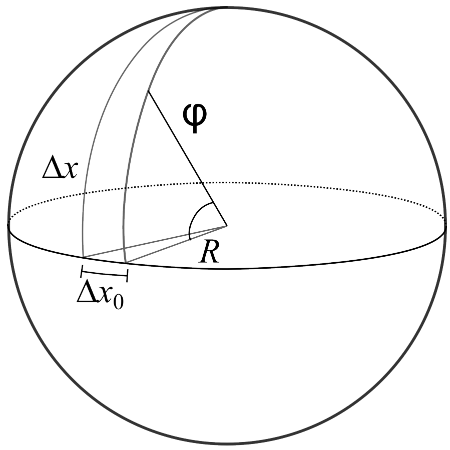

Figure 6:

Two people walking on "great circles" converge.

$$\begin{aligned}\Delta x &= \Delta x_0 \cos\phi \\ &= \Delta x_0 \cos\left(\tfrac{S}{R}\right)\end{aligned}$$

$$\begin{aligned}\therefore \tfrac{d^2}{ds^2}\Delta x &= -\tfrac{\Delta x_0}{R^2}\cos\left(\tfrac{S}{R}\right) \\ &= -\tfrac{1}{R^2}\Delta x\end{aligned}$$

Compare this $\ddot{x}^i - K^i{}_j x^j \quad K^i{}_j = \partial^i \partial_j \phi$ to the "tidal equation"

The trajectories of the people on the sphere mimic the trajectories of particles in a gravitational field.

$$K \sim \frac{1}{R^2}$$

Partial Trajectory $\sim$ Motion on great circle $\sim$ shortest distance

Note $\nabla^2 \sim \rho_\mu$ so $\frac{1}{R^2}$ should relate to mass in some way.

This is the idea of general relativity.

Particles move on paths of shortest distance "Geodesics" e.g. Great Circles

Spacetime is curved

Curvature is caused by mass.

We think of the "curved" trajectories as gravitational force.

More on curved spaces

Need an intrinsic notion of curvature.

This is, we need to define the curvature of a space without refering to the space being embedded in something else.

Consider the surface of a ball

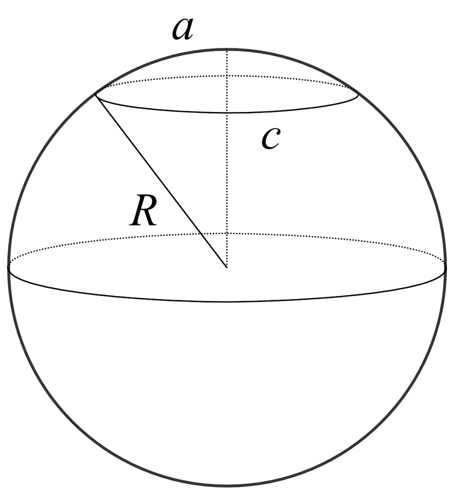

Figure 7:

Circle Circumference is:

$$c = 2\pi R \sin\left(\frac{a}{R}\right)$$

$$K = \frac{1}{R^2}$$

$$a \to 0, \quad c \to 2\pi R\left(\frac{a}{R} - \frac{1}{6}\left(\frac{a}{R}\right)^3\right) = 2\pi a\left(1 - \underbrace{\frac{1}{6}Ka^2}_{\text{Flat Space Correction}}\right)$$

$$K \equiv \frac{3}{\pi} \lim_{a \to 0} \left(\frac{2\pi a - c}{a^3} \right)$$

This definition works for any surface without mentioning an embedding space & is local i.e. the curvature need not be constant over the surface.

$K(x)$ is the Gaussian Curvature

$K > 0$ e.g. sphere

$K = 0$ e.g. plane

$K < 0$ e.g. Saddle point

Equation of geodesic derivation $\frac{d^2 g}{ds^2} = -Kg$ on any surface

$K > 0$ e.g. converges

$K = 0$ e.g. diverges

$K < 0$ e.g. straight

Note: K is unchanged under bending that is deformations that don't bend or stretch.

Think of a sphere & a cylinder

A cylinder is not curved.

K is local, it depends where you are.

To define K we need a length on the surface.

Let first look at a simple 2-surface in 3-d to see how things work.

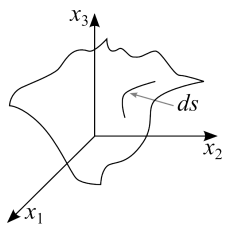

Figure 8:

$$\underbrace{x_3 = x_3(x_1, x_2)}_{\text{Defines curved 2-surface embedded in 3-d.}}$$

Define

$$\frac{\partial x^i}{\partial x'^j} = \Lambda_j{}^i$$

Then

$$g'_{lm} = \Lambda_l{}^i \Lambda_m{}^k g_{ik}$$

$$
\begin{aligned}
ds^2 &= dx_1^2 + dx_2^2 + dx_3^2 \\
dx_3 &= \tfrac{\partial x_3}{\partial x_1} dx_1 + \tfrac{\partial x_3}{\partial x_2} dx_2
\end{aligned}
$$

$$
\begin{aligned}
ds^2 &= \left(1 + \left(\tfrac{\partial x_3}{\partial x_1}\right)^2\right) dx_1^2 + \left(1 + \left(\tfrac{\partial x_3}{\partial x_2}\right)^2\right) dx_2^2 + 2 \tfrac{\partial x_3}{\partial x_2} \tfrac{\partial x_3}{\partial x_1} dx_1 dx_2 \\
&\equiv \underbrace{g_{ij} dx^i dx^j}_{\substack{\text{Metric a function} \\ \text{of } x_1 \text{ \& } x_2\text{. No ref} \\ \text{to } x_3 \text{ anymore.}}} \quad g_{ij}(x_1, x_2)
\end{aligned}
$$

New Coordinates$x^{i\prime} = x^{i\prime}(x)$ $\quad \left| \begin{aligned} x^{1\prime} &= x^{1\prime}(x^1, x^2) \\ x^{2\prime} &= x^{2\prime}(x^1, x^2) \end{aligned} \right.$

$$
\begin{aligned}
ds^2 &= g_{ij} dx^i dx^j \\
&= g_{ij}(x') \tfrac{\partial x^i}{\partial x'^l} dx^{l\prime} \tfrac{\partial x^j}{\partial x'^m} dx'^m \\
&= g'_{lm}(x') dx'^l dx'^m \\
g'_{lm}(x') &= g_{ij} \tfrac{\partial x^i}{\partial x'^l} \tfrac{\partial x^j}{\partial x'^m}
\end{aligned}
$$

Metric describes the intrinsic geometry but does depend on our choice of coordinates. E.G. R^2, flat space with Cartesian vs Polar Coordinates.

This will be a constant concern.

Our choice of coordinates cannot matter and so we must understand things in a non-coordinate-dependent way, where we can.

Other examples in physics, gauge theories

$$B = \mathrm{curl}(A) \qquad\qquad A \to A + \nabla x \quad \text{etc.}$$

We know we can always find coods, such that at any chosen point, P

$$g_{ij}|_P = \delta_{ij}$$

(This is like a freely falling frame.) If we can do this everywhere then the space is flat. If it is not possible then the space is reserved.

<u>Question</u>

Given $g_{ij}(x)$ & $g_{ij}(x')$

when do they define the same geometry?

Simple question, when is space flat?

If $K = 0$ everywhere then it is flat & coords can be found such that $ds^2 = dx^2 + dy^2$

Gauss Showed

(use $g_{12} = 0$)

$$K = \frac{1}{g_{11}g_{22}} \left(\frac{\partial^2 g_{11}}{\partial (x^2)^2} - \frac{\partial^2 g_{22}}{\partial (x^1)^2} + \frac{1}{2g_{11}} \left[\frac{\partial g_{11}}{\partial x^1} \frac{\partial g_{22}}{\partial x^2} \right] + \frac{1}{2g_{22}} \left[\frac{\partial g_{11}}{\partial x^2} \frac{\partial g_{22}}{\partial x^1} + \left(\frac{\partial g_{22}}{\partial x^1} \right)^2 \right] \right)$$

This is for a curved 2-surface.

We want a four surface & with a time direction.

Definition Riemannian Space

An n-dimensional space with n arbitrary coords. x^i whose intrinsic geometry is defined by a metric:

$$ds^2 = \sum_{i \cdot j=1}^{n} g_{ij}(x) dx^i dx^j$$

(Usually locally Euclidean i.e. $g_{Ij}|_p = \delta_{ij}$)

We want locally Minkowski i.e. $g_{ij}|_p = \eta_{ij}$

Use greek indices from now on for Minkowski signature.

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu}(x) dx^\mu dx^\nu$$

$$\& \quad g_{\mu\nu}|_p = \eta_{\mu\nu}$$

i.e. Can find coords. at a point such that metric is Minkowski.

i.e. $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$

or in rel. conventions $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$

Physical Significance

Spacetime is a 4-d Riemannian Space with geometry described by the metric

$$ds^2 = \underbrace{\sum_{\mu,\nu} g_{\mu\nu}(x) dx^\mu dx^\mu}_{\text{ten indep. functions}}$$

Locally coords. may be found such that

$$g_{\mu\nu}|_P = \eta_{\mu\nu}$$

This is a freely falling frame.

If coords can be found such that Equation everywhere then the space is flat & special relativity holds.

Else the space is curved and so there is gravity.

Gravity is spacetime curvature!

Physics	Maths
Locally Inertial Frame at a point.	Coords found such
S.R. holds & gravity can be	that $g_{\mu\nu}|_P = 0$
locally ignored	$\frac{\partial g_{\mu\nu}}{\partial x^2}|_P = 0$
Still Tidal forces	Curvature
K_{ij}	

We will need a generalization of curvature to four dimensions.

It will again be 2nd derivatives of $g_{\mu\nu}$ & non-linear.

Idea:

i) Find how matter/energy curved space.

ii) Find how curvature of space effects physics.

Tensors

Principle of General Covariance

The general laws of physics can be stated in a form which is independent of the choice of spacetime coordinates.

Use tensor to write down laws that are automatically covariant under coord. change.

Invariance - things stay the same.

Covariance - the form stays the same.

Start in 3-d with a vector equation

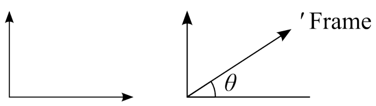

Figure 9:

$$\begin{array}{rrcl} & \vec{A} & = & \vec{B} \\ \Rightarrow & A_i & = & B_i \\ \Rightarrow & A_i' & = & B_i' \\ \Rightarrow & & & \text{covariant equation} \end{array}$$

in components after rotating frame

In curved space we need to work locally.

Take $dx^\mu|_P = x^\mu|_Q - x^\mu|_P$

With Q infinitesimally close to P.

How does dx^μ transform under $x \to x'(x)$

$$\begin{aligned} dx^\mu &= \sum_\nu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu \\ &= \sum_\nu \Lambda_\mu{}^\nu dx'^\nu \end{aligned}$$

with $\Lambda_\mu{}^\nu = \frac{\partial x^\mu}{\partial x'^\nu}$

$$
\begin{aligned}
&\Lambda_\sigma{}^\nu \\
\frac{\partial}{\partial \bar{x}^\sigma} &= \frac{\partial x^\nu}{\partial \bar{x}^\sigma}\frac{\partial}{\partial x^\nu} \\
\partial_{\sigma'} &= \Lambda_\sigma{}^\nu \partial_\nu
\end{aligned}
$$

$$
\begin{aligned}
d\bar{x}^\nu &= \frac{\partial \bar{x}^\nu}{\partial x^\mu} dx^\mu \\
d\bar{x}^\nu &= \Lambda^\nu{}_\mu dx^\mu
\end{aligned}
$$

$$\begin{aligned}
dx'^{\mu} &= \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu} \\
&= \Lambda^{\mu}{}_{\nu} dx^{\nu} \\
\Lambda^{\mu}{}_{\nu} \underbrace{\Lambda_{\sigma}{}^{\nu}}_{\substack{\text{defines}\\ \text{invariance}\\ \text{of } \Lambda^{\mu}_{\nu}}} &= \delta_{\sigma}{}^{\mu}
\end{aligned}$$

Since

$$\begin{aligned}
dx'^{\mu} &= \Lambda^{\mu}{}_{\nu} \Lambda_{\sigma}{}^{\nu} dx'^{\sigma} \\
&= \delta^{\mu}_{\sigma} dx'^{\sigma}
\end{aligned}$$

A note on Kronecker delta. $\delta^{b}_{a} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$

$\Lambda_{\sigma}{}^{\nu}$ is a covariant transformation

$\Lambda^{\mu}{}_{\nu}$ is a contravariant transformation

e.g.

$$\begin{aligned}
g' &= x \sin\theta + y \cos\theta \\
x' &= x \cos\theta - y \sin\theta
\end{aligned}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore x'^{i} = R^{i}{}_{j} x^{j} \qquad \text{- contravariant}$$

Contravariant means it transforms like a vector would.

Basically

- upper index is contra
- lower index is covariant

Definition

A vector field. Given four functions $A^\mu(x)$, we say these are components of a vector field if at every point the A^μ transform contravariantly.

$$A'^\mu = \Lambda^\mu{}_\nu(x) A^\nu$$

A scalar field is invariant

i.e. $\phi(x) \to \phi(x')$

How to derivatives transform?

Take derivative of a scalar field:

$$\begin{aligned} \frac{\partial \varphi'(x')}{\partial x'^\mu} &= \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} \underbrace{\varphi'(x')}_{=\varphi(x)} \\ &= \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} \varphi(x) \\ &= \underbrace{\Lambda_\mu{}^\nu \partial_\nu \varphi}_{\text{covariant}} \end{aligned}$$

Recall $\Lambda_\mu{}^\nu$ is the inverse of $\Lambda^\mu{}_\nu$

$$B'_\mu(x') = \Lambda_\mu{}^\nu B_\nu(x)$$

covariant vector field

Note: $A^\mu(x)B_\mu(x)$ is a scalar.

Need a "free" index to transform.

Higher Rank Tensors

We have already met the metric

$$g_{\mu\nu}(x') = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma g_{\rho\sigma}$$

For each free index use a $\Lambda_\mu{}^\rho$ if index is lower or $\Lambda^\mu{}_\rho$ if index is upper.

The number of free indices is the rank of the tensor.

The metric is a rank two covariant tensor.

Can have mixed co & contravariant tensors.

$$T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}(x)$$

$$\to \Lambda^{\mu_1}{}_{\rho_1}\ldots\Lambda^{\mu_r}{}_{\rho_r}\Lambda_{\nu_1}{}^{\sigma_1}\ldots\Lambda_{\nu_s}{}^{\sigma_s}T^{\rho_1\ldots\rho_r}{}_{\sigma_1\ldots\sigma_s}(x)$$

This is a mixed tensor of rank $r+s$.

Note: the transformation is linear and homogeneous.

The most important tensor is the zero tensor.

It is always zero.

So if $S^{\mu\ldots}{}_{\rho\ldots} = T^{\mu\ldots}{}_{\rho\ldots}$

then $S - T = 0$ is true for all frames A covariant equation.

Properties of Tensors

Sum of tensors of one kind are the same kind.

$$C_\mu = A_\mu + B_\mu$$

If A_μ & B_μ are covariant tensors then so is C_μ.

Product of tensors is a tensor with the rank equal to the sum of the ranks.

ie.

$$A^{\overbrace{\sigma_1 \ldots \sigma_r}^{r}} + B_{\underbrace{\rho_1 \ldots \rho_s}_{s}} = \underbrace{T^{\sigma_1 \ldots \sigma_r}{}_{\rho_1 \ldots \rho_s}}_{\text{rank } r+s}$$

Contraction of Indices

So:

- $A^\mu B_\mu$ is scalar
- $T^\mu{}_\mu$ is also scalar
- $T^\mu{}_\mu$ is the contraction of $T^\mu{}_\nu$ i.e. $\sum_\mu T^\mu{}_\mu$

Obviously generalises

- $T^{\nu\mu}{}_\mu$ is a contravariant 1 tensor etc.
- $T_{\mu\nu}{}^\nu{}_\rho{}^\rho$ is a covariant 1 tensor.

Just look at the "free" index.

Raising & Lowering Indices

How can we turn a covariant (lower) index into a contravariant (upper) index? Use the metric or it's inverse.

$$g^{\mu\nu} \equiv \left(g^{-1}\right)^{\mu\nu}$$

the inverse of the metric.

So

$$T_\mu = g_{\mu\nu} T^\nu$$
$$T^\mu = g^{\mu\nu} T_\nu$$
$$T^\mu{}_\mu = T^{\mu\nu} g_{\mu\nu} = T_{\mu\nu} g^{\mu\nu}$$
$$g_{\mu\rho} g^{\nu\sigma} = \delta_\mu{}^\nu$$

<u>Tensor Calculus</u>

$\partial_\mu \varphi$ derivative of a scalar covariant tensor.

$$\partial_\mu A_\nu = ?$$

$$A'_\rho(x') = \Lambda_\rho{}^\mu(x) A_\mu(x)$$

$$\begin{aligned} \partial'_\sigma A'_\rho(x') &= \partial'_\sigma \left(\Lambda_\rho{}^\mu A_\mu(x) \right) \\ &= \Lambda_\rho{}^\mu \partial'_\sigma A_\mu + \left(\partial'_\sigma \Lambda_\rho{}^\mu \right) A_\mu \end{aligned}$$

$$\partial'_\sigma = \Lambda_\sigma{}^\nu \partial_\nu$$

$$\therefore \partial'_\sigma A'_\rho(x') = \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu \partial_\nu A_\mu + \left(\Lambda_\sigma{}^\nu \partial_\nu \Lambda_\rho{}^\mu \right) A_\mu$$

Recall $\Lambda_\sigma{}^\nu = \frac{\partial x^\nu}{\partial x'^\sigma}$

$$\frac{\partial x^\nu}{\partial x'^\sigma} \frac{\partial}{\partial x^\nu} \frac{\partial x^\mu}{\partial x'^\rho} = \frac{\partial^2 x^\mu}{\partial x'^\sigma \partial x'^\rho}$$

This is linear but <u>not</u> homogeneous.

Not a tensor!

Want a derivative that transforms covariantly.

Define the covariant derivative:

$$D_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\alpha{}_{\nu\mu} A_\alpha$$

$\Gamma^\alpha{}_{\nu\mu}$ is called the connection.

Under $x \to x'(x)$

$$\Gamma \to \Gamma'$$
$$\Gamma'^\alpha{}_{\beta\gamma} = \Lambda^\alpha{}_\mu \Lambda_\beta{}^\nu \Lambda_\gamma{}^\rho \Gamma^\mu{}_{\nu\rho} + \Lambda_\sigma{}^\alpha \frac{\partial^2 x^\sigma}{\partial x'^\beta \partial x'^\gamma}$$

Also not a tensor!

Idea is to add to non tensors to cancel out the problem term to create a tensor.

Then

$$D'_\sigma A'_\rho = \Lambda_\sigma{}^\nu \Lambda_\rho{}^\mu \left(D_\nu A_\mu\right)$$

<u>Properties</u>

1. Distributive: $\partial_x(f_g) = (\partial_x f)_g + f\partial_x g$

$$D_\mu \underbrace{(S \quad T)}_{\substack{\text{tensors} \\ \text{any rank}}} = (D_\mu S)T + S(D_\mu T)$$

2. $D_\mu \varphi = \partial_\mu \varphi$

For a covariant tensor

$$D_\mu B^\nu = \partial_\mu B^\nu + \Gamma^\nu{}_{\mu\rho} B^\rho$$

Can prove this by $D_\mu(A_\nu B^\nu) = \partial_\mu(A_\nu B^\nu)$
For a general tensor

$$D_\rho T^{\mu\dots} = \partial_\rho T^{\mu\dots}{}_{\nu\dots} + \Gamma^{\mu_i}{}_{\rho\beta} T^{\mu_1..\beta..}{}_{\nu..} - \Gamma^\beta{}_{\rho\nu_i} T^{\mu\dots}{}_{\nu\beta\dots}$$

Want covariant differentiation to commute with a raising & lowering.

$$\begin{gathered} D_\rho(A_\mu) = g_{\mu\nu}(D_\rho A^\nu) \\ = D_\rho(g_{\mu\nu}A^\nu) = D_\rho(g_{\mu\nu})A^\nu + g_{\mu\nu}(D_\rho A^\nu) \end{gathered}$$

for this to be true. That is the covariant derivative of the metric must vanish!
This determines Γ in terms of the metric g.

$$\begin{aligned} &1) \Rightarrow \partial_\rho g_{\mu\nu} - g_{\mu\alpha}\Gamma^\alpha{}_{\rho\nu} - g_{\alpha\nu}\Gamma^\alpha{}_{\mu\rho} = 0 \\ &\Gamma^\gamma{}_{\mu\alpha} = \frac{1}{2} g^{\gamma\nu}(\partial_\alpha g_{\nu\mu} + \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\mu\alpha}) \end{aligned}$$

Called Christoffel symbols

- or metric compatible connection.
- or Riemann connection

Gives Riemann Geometry.

Natural Coordinates

Want to prove that coordinates can be chosen (at a point) such that:

$$\frac{\partial g_{\mu\nu}}{\partial x^\alpha} = 0$$

(Assume Equation $g_{\mu\nu}|_\mu = \eta_{\mu\nu}$)

This is equivalent to $\Gamma^\alpha{}_{\mu\nu}|_\mu = 0$ since $D_\rho g_{\mu\nu} = 0$

Find coords such that $g_{\mu\nu}|_\mu = \eta_{\mu\nu}$

but $\Gamma^\alpha{}_{\mu\nu}|_\mu = 0$

$g_{\mu\nu}$ transforms linearly $g'_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta g_{\alpha\beta}$

but $\Gamma'^\alpha{}_{\beta\gamma} = (\Lambda^{-1})^\alpha{}_\sigma \Lambda_\beta{}^\mu \Lambda_\gamma{}^\nu \Gamma^\sigma{}_\mu + \Lambda_\sigma{}^\alpha \frac{\partial^2 x^\sigma}{\partial x^\beta \partial x'^\gamma}$

Choose $x^\sigma(x'^\tau)$ to cancel between the two terms.

Take origin at P. $x' = 0$ & $x = 0$ at P.

Try $x^\sigma = x'^\sigma = \frac{1}{2}\Gamma^\alpha{}_{\alpha\beta}|_\rho x'^\alpha x'^\beta$

Check $\Lambda_\mu{}^\sigma = \frac{\partial x^\sigma}{\partial x'^\mu} = \delta_\mu{}^\sigma = \Gamma^\sigma{}_{\mu\beta} x'^\beta$

$$\therefore \Lambda_\mu{}^\sigma|_\rho = \delta_\mu{}^\sigma$$

$$\therefore g'|_\rho = \eta$$

$$\frac{\partial^2 x^\sigma}{\partial x'^\nu \partial x'^\mu}|_\rho = -\Gamma^\sigma{}_{\mu\nu}$$

$$\therefore \Gamma'{}_{\beta\gamma}|_\rho = 0$$

These coords correspond to the freely falling frame.

Principle of General Covariance Again

Laws of physics are generally covariant and can be built in tensor form.

Add principle of equivalence; in natural coords these laws are special relativity.

So write down laws which are special relativistic in natural coords & then covariantize by introducing covariant derivative.

e.g.

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$$
$$\partial^\mu\partial_\mu\phi = 0$$

Just replace ∂_μ with D_μ

Since $\Gamma = 0$ & $g = \eta$ in natural coords it is the same equation in those coords but with the time now for any coords.

Equations generally covariant.

$$\rightarrow D_\mu D^\mu\phi = 0 \qquad \text{generally covariant.}$$

Key Example: Geodesics

Mechanics of free particles

Consider a curve

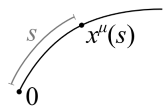

Figure 10:

$U^\mu = \frac{dx^\mu}{ds}$ is the tangent vector to the curve

This gives the direction of the particle travelling along the curve.

Free particles travel in straight lines. This means:s

$$\frac{d}{ds}U^\mu = 0$$
$$\frac{d}{ds}\frac{dx^\mu}{ds} = 0$$

Straight line means the tangent vector is unchanging.

If we pick S - the coord. that parameterise the curve to be x^0 - time, then

$$\frac{d^2x^i}{dt^2} = 0 \qquad \text{Newton's Law of Motion of particles in flat space.}$$

Covariantize this to give motion of particle in curved space.

$$\frac{d}{ds}|_{\text{path}} = \frac{\partial x^\nu}{\partial s}\frac{\partial}{\partial x^\nu}|_{\text{path}} = U^\nu \frac{\partial}{\partial x^\nu}$$
$$\frac{d}{ds}U^\mu = U^\nu \frac{\partial}{\partial x^\nu}U^\mu = U^\nu D_\nu U^\mu \qquad \text{for flat and curved space.}$$

$$U^\nu D_\nu U^\mu = 0 \qquad \text{for a straight line in curved space.}$$

$$U^\nu \left(\partial_\nu U^\mu + \Gamma^\mu{}_{\nu\alpha}U^\alpha\right) = 0$$

$$\Rightarrow \frac{d^2}{ds^2}x^\mu + \Gamma^\mu{}_{\nu\alpha}\frac{dx^\nu}{ds}\frac{dx^\alpha}{ds} = 0$$

"Geodesic Equations"

Geodesic is defined as shortest distance between two points

$$L = \int_P^Q ds = \int_P^Q \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda$$

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= g_{\mu\nu}\tfrac{dx^\mu}{d\lambda}d\lambda\tfrac{dx^\nu}{d\lambda}d\lambda \end{aligned}$$

$$ds = \sqrt{ds^2} = \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda$$

$$\partial L = 0 \Rightarrow \text{Minimize } \ L$$

$\Rightarrow$ Euler-Lagrange Equations $\Rightarrow$ "Geodesic Equation"

<u>Riemannian Curvature</u>

Consider

$$D_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\alpha{}_{\nu\mu}A_\alpha$$

then

$$\begin{aligned} D_\sigma(D_\nu A_\mu) &= \partial_\sigma D_\nu A_\mu - \Gamma^\alpha{}_{\sigma\nu}D_\alpha A_\alpha - \Gamma^\alpha{}_{\alpha\mu}D_\nu A_\alpha \\ &\equiv D_\sigma D_\nu A_\mu \end{aligned}$$

$$(D_\sigma D_\nu - D_\nu D_\sigma)A_\mu = A_\varepsilon R^\varepsilon{}_{\mu\nu\sigma}$$

$$R^\varepsilon{}_{\mu\nu\sigma} = \partial_\sigma \Gamma^\varepsilon{}_{\mu\nu} - \partial_\nu \Gamma^\varepsilon{}_{\mu\sigma} + \Gamma^\alpha{}_{\mu\sigma}\Gamma^\varepsilon{}_{\alpha\nu} - \Gamma^\alpha{}_{\mu\nu}\Gamma^\varepsilon{}_{\alpha\sigma}$$

$R^{\varepsilon}{}_{\mu\nu\sigma}$ is the Riemann Curvature tensor.

Analogy between Relativity & G.R.:
A_μ vector pot! in EMF
$\Gamma^{\varepsilon}{}_{\mu\nu}$ connection in G.R.
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in EMF
$R_{\mu\nu\rho\sigma} = \ldots$ in G.R.

Properties of $R^{\varepsilon}{}_{\mu\nu\sigma}$

$R = 0 \quad \Rightarrow \quad$ spacetime is flat

Depends on 2nd derivatives like Gaussian Curvature

$$R^{\varepsilon}{}_{\mu\nu\sigma} = -R^{\varepsilon}{}_{\mu\sigma\nu}$$

$$R^{\varepsilon}{}_{\mu\nu\sigma} + R^{\varepsilon}{}_{\nu\sigma\mu} + R^{\varepsilon}{}_{\sigma\mu\nu} = 0$$

$$R_{\rho\mu\nu\sigma} = g_{\rho\varepsilon} R^{\varepsilon}{}_{\mu\nu\sigma}$$

$$R_{\rho\mu\nu\sigma} = -R_{\mu\rho\nu\sigma}$$

$$R_{\rho\mu\nu\sigma} = R_{\nu\sigma\rho\mu}$$

$\Rightarrow$ in n dimensions $\frac{n^2(n^2-1)}{12}$ indep. variables.

$n = 4$	20	variables
$n = 2$	1	variable $= K$ the Gaussian Curvature

Equation of Geodesic Deviation

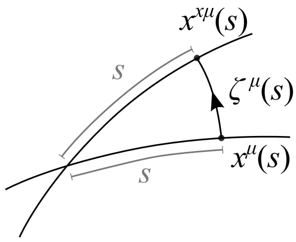

Figure 11:

$$x^{*\mu}(s) = x^{\mu}(s) + \zeta^{\mu}(s)$$
$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}{}_{\nu\alpha}\frac{dx^{\nu}}{ds}\frac{dx^{\alpha}}{ds} = 0$$
$$\therefore \frac{d^2}{ds^2}(x^{\mu} + \zeta^{\mu}) + \Gamma^{\mu}{}_{\nu\alpha}(x+y)\frac{d}{ds}(x^{\nu} + \zeta^{\nu})\frac{d}{ds}(x^{\alpha} + \zeta^{\alpha}) = 0$$

Taylor Expand $\Rightarrow \frac{D^2}{DS^2}\zeta^{\mu} = -R^{\mu}{}_{\nu\rho\alpha}\zeta^{\rho}\frac{dx^{\nu}}{ds}\frac{dx^{\alpha}}{ds}$

N.B. $\frac{D}{DS} = \frac{dx^{\mu}}{ds}D_{\mu}$

Relative Acceleration of neighbouring Geodesics.

If $R^{\mu}{}_{\nu\rho\alpha} = 0$ then there is no acceleration & geodesics remain equidistant.

e.g. A sphere $R^{\phi}{}_{\theta\phi\theta} = \frac{1}{R^2}$ and $s = \theta$

So
$$\frac{D^2}{D\theta^2}\zeta^{\phi} = -\frac{1}{R^2}\zeta^{\phi}$$

Looks like a SHO.

Einsteins Field Equations

Compare Geodesic Equation to Newtons Tidal Equation.

$$\frac{d^2}{dt^2}\zeta_i = -k_{ij}\zeta^j$$
$$k_{ij} = \partial_i\partial_j\phi$$
$$k_{ij} \approx R^{\mu}{}_{\nu\rho\alpha}\frac{dx^\mu}{ds}\frac{dx^\alpha}{ds}$$
$$k_{ij} = \nabla^2\phi = +4\pi G\rho = 0 \text{in vacuum}$$

Suggests $R^{\mu}{}_{\nu\rho\alpha}\frac{dx^\mu}{ds}\frac{dx^\alpha}{ds} = 0$ in vacuum

In Fact, The vacuum Field Equation:

$$R^{\mu}{}_{\nu\mu\alpha} = 0$$

Ricci Tensor $R_{\nu\alpha} = R^{\mu}{}_{\nu\mu\alpha}$

Einstein Field Equation. (Vacuum)

$$R_{\mu\nu} = 0$$

Second Order Equation for $g_{\mu\nu}$, actually 10 equations.

Flat space is a solution as it should be.

Can Generalise this: $\underbrace{R_{\mu\nu} + \lambda g_{\mu\nu}}_{\lambda=\text{Cosmological Const}}$

Flat Space not a solution.

Solutions to this are called Einstein Spaces.

They are spaces of constant curvature.

In fact the vacuum of our universe appears to have $\lambda < 0$

This is called "de Sitter Space".

λ is very small but on current data non-zero.

What is this constant?

There are also studies of $\lambda > 0$ which is "anti de Sitter" space.

<u>Newtons Theory as an approximation</u>

Should recover Newtons laws of gravitation for work fields.

$$\Rightarrow g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{where } h_{\mu\nu} \text{ is small.}$$

Also Newtons theory is static so $\frac{\partial}{\partial t} h_{\mu\nu} = 0$

Substitute into

$$R_{\mu\nu} = 0$$

To make life easy $h_{\mu\nu} = 2\delta_{\mu\nu} V(\bar{x})$

$$\therefore ds^2 = (\eta_{\mu\nu} + 2V(x)\delta_{\mu\nu}) dx^\mu dx^\nu$$

Can show $R_{\mu\nu} = -\delta_{\mu\nu} \nabla^2 V$

$$\nabla^2 = \partial_i \partial^i$$

This suggests that $V(\bar{x})$ is the Potential $\phi(x)$

$\phi =$ -V ϕ is the Gravitational Pot!

Go Back To $R^\mu{}_{\nu\rho\mu} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} = +k_{ij} = +4\pi G\rho$

Newtons laws are non relativistic so can

Choose $s = t \Rightarrow \frac{dx^\nu}{ds} = (1, 0, 0, 0)$

$$\therefore R_{\gamma\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = R_{tt} = -\nabla^2 V = \nabla^2\phi$$

&

$$-\nabla^2 V = +4\pi G\rho$$
$$\Rightarrow R_{tt} = 4\pi G\rho$$

Relation between Curvature & Matter.

Look at Geodesics

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$$
$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta})$$
$$\begin{array}{rcl} \Rightarrow \frac{d^2x^i}{ds^2} &=& -\Gamma^i{}_{00}\text{using} \quad \frac{dx^\mu}{ds} = (1,0,0,0) \\ \therefore \quad &=& -\frac{1}{2}\partial_i h_{00} \\ &=& -\nabla_i\phi \quad \text{if} \quad \frac{1}{2}h_{00} \end{array}$$

All this leads to $h_{\mu\nu} = 2\delta_{\mu\nu}\underbrace{\phi}_{\substack{\text{grav}\\\text{pot!}}}$

This is a weak field, static approximation.

eg A central body with mass M:

$$\phi = -\frac{GM}{r}$$
$$\begin{array}{rcccccc} ds^2 &=& -(1-2\frac{GM}{r})dt^2 &+& (1+\frac{GM}{r})dr^2 &+& r^2 d\Omega^2_{(2)} \\ &=& -(1+2\phi)dt &+& (+1+2\phi)dr^2 &+& r^2 d\Omega^2_{(2)} \end{array}$$

Approximate from good for Newtonian Approximation.

Beyond the vacuum, so far we have

$$R_{tt} = 4\pi G R$$

Want to Relativise this.

$R_{\mu\nu} =$? 2nd rank tensor whose 0 0 components are $4\pi G\rho$

This is called the energy momentum tensor.

<u>Summary Vacuum</u>:

	Einstein	Newton
Vacuum Field Equ	$R_{\mu\nu} = 0$	$\nabla^2\phi = 0$
Particle	Geodesic Motion	$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\phi$
Matter	$R^{\mu}{}_{\rho\sigma\mu}u^{\rho}u^{\sigma}$	K_{ii}

Covariantize

$$G_{\mu\nu} = T_{\mu\nu}$$

Stress Energy Tensor

$$T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{0i} \\ T^{i0} & T^{ij} \end{pmatrix}$$

$$\begin{aligned} T^{00} &= \text{Mass/Energy Density} \\ T^{0i} &= \text{Energy Flux} \\ T^{i0} &= \text{Moment Density} \\ T^{ij} &= \text{Stress eg Pressure} \end{aligned}$$

Eg Electromagnetion

$$\begin{aligned} T^{00} &= \tfrac{1}{2}(|E|^2 + |B|^2) \\ T^{0i} &= (E \times B)_i \end{aligned}$$

Look at Newton Approx

$$\begin{aligned} R_{\mu\nu} &= +\delta_{\mu\nu}\nabla^2\phi \\ R_{\mu\nu} &= +\delta_{\mu\nu}\nabla^2\phi + O(h) = 4\pi G \begin{pmatrix} \rho & & & \\ & \rho & & \\ & & \rho & \\ & & & \rho \end{pmatrix} \end{aligned}$$

Case: Static & no stress $\rho(x)$ $\quad T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

Connect $R^{\mu\nu}$ & $T^{\mu\nu}$

Recall $\eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$

So $\eta^{\mu\nu}\rho 4\pi G = 4\pi G \begin{pmatrix} -\rho & & & \\ & +\rho & & \\ & & +\rho & \\ & & & +\rho \end{pmatrix}$

So $R^{\mu\nu} - 4\pi G\rho\eta^{\mu\nu} = 4\pi G \quad 2T^{\mu\nu} = 8\pi G T^{\mu\nu}$

$$R^{\mu}{}_{\nu} = +4\pi G \begin{pmatrix} -\rho & & & \\ & +\rho & & \\ & & +\rho & \\ & & & +\rho \end{pmatrix}$$

$$R \equiv R^{\mu}{}_{\mu} = 4\pi G 2\rho \quad \text{Scalar Curvature}$$

$$\therefore R^{\mu\nu} - \frac{1}{2} R \eta^{\mu\nu} = 8\pi G T^{\mu\nu}$$

$$\therefore R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = 8\pi G T^{\mu\nu}$$

& with Cosmological Const.

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \lambda\, g^{\mu\nu} = 8\pi G T^{\mu\nu}$$

Einstein Equation

In CGS units $\frac{8\pi G}{c^4}$

Check Vacuum

$$\begin{array}{lcl} T^{\mu\nu} & = & 0 \\ \lambda & = & 0 \end{array}$$

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = 0$$

$$g_{\mu\nu}(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}) = 0$$

$$R - \frac{1}{2} R 4 = 0$$

$$\Rightarrow R = 0$$

Iden is given $T^{\mu\nu}$ then find $R^{\mu\nu}(g)$. Stress Energy sources spacetime curvature. The Equations are very hard to solve. Non linear Coupled differential Equations. Non-linear $\Rightarrow$ No superposition principle.

Weak Field Approx: $g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{h_{\mu\nu}}_{\text{small}}$

$$\Box h_{\mu\nu} = \underbrace{H}_{\frac{8\pi G}{c^4}} T_{\mu\nu}$$

Compare with EM,

$\Box A_\mu = j_\mu$ Wave Equation with current source

$\Box h_{\mu\nu} = 0$ Wave Equation

$\Rightarrow$ Gravitational waves, stretching of spacetime. (Be careful of Coordinate changes just like $A_\mu + A_\mu + \partial_\mu \chi$)

<u>Problem with Einstein Equations</u>

10 equs for 10 unknowns $g_{\mu\nu}$

But Coord. trans. $\Rightarrow g \to g'(x') = \wedge \wedge g(x)$

So there arent really 10 unknowns since coord changes are the same physical space.

$\Rightarrow$ Must be some identities in Einstein Equations

We will find these,

Look at $T_{\mu\nu}$

1. Blob of Static Dust in frame where dust is at rest.

$$\begin{aligned} T^{\mu\nu} &= \left(\begin{array}{c|c} \rho & 0 \\ \hline 0 & 0 \end{array}\right) \\ T^{\mu\nu} &= \rho(x)\tfrac{dx^\mu}{ds}\tfrac{dx^\nu}{ds} \qquad \text{in any frame} \end{aligned}$$

   Since in Static frame $x^0 = s$

   & then if time in one frame its time in all frames.

2. Perfect fluid in frame where fluid is at rest & pressure is isotropic.

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & \rho & & \\ & & \rho & \\ & & & \rho \end{pmatrix}$$

   Convariantize: $T^{\mu\nu} = (\rho + \eta)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + g^{\mu\nu}\eta$

Properties of $T^{\mu\nu}$

In Flat Space $\partial_\nu T^{\mu\nu} = 0$

Conservaion of Energy & Momentum.

$\underline{\mu = 0 \text{ Equ}}$

$$\begin{gathered} \partial_\nu T^{0\nu} = 0 \\ = \partial_0 T^{00} + \partial_i T^{0i} = 0 \end{gathered}$$

Integrate over Volume bounded by a surface s:

$$\int_V \partial_0 T^{00} dV + \int_V \partial_i T^{0i} dV = 0$$

Divergence Theorem

$$\underbrace{\frac{d}{dt}\left(\int_V T^{00} dV\right)}_{E} + \int_S T^{0i} dS_i$$

$$\therefore \frac{d}{dt} E = -\int_S \underbrace{T^{0i}}_{\substack{\text{Energy}\\ \text{Flux}}} dS_i$$

$\underline{\mu = I}$ $\Rightarrow$ Conservation of momentum

Covariantize

$$\partial_\mu T^{\mu\nu} = D_\mu T^{\mu\nu} = 0$$

$$D_\mu T^{\mu\nu} = 0$$

So $\Rightarrow$ for Einsteins Equs

$$D_\nu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \lambda g^{\mu\nu})$$

This is a geometrical identity.

4 equations

$\Rightarrow$ Only 6 indep. Einstein Equations.

<u>Proof:</u> Bianchi Identity

$$R^{\varepsilon}{}_{\mu\nu\sigma} = \partial_\nu \Gamma^{\varepsilon}{}_{\mu\sigma} - \partial_\sigma \Gamma^{\varepsilon}{}_{\mu\nu} + \Gamma\Gamma - \Gamma\Gamma$$

Diff wrt ρ covariantly & use natural cords such that $\Gamma = 0$

$$D_\rho R^{\varepsilon}{}_{\mu\nu\sigma} = \partial_\rho \partial_\mu \Gamma^{\varepsilon}{}_{\mu\sigma} - \partial_\rho \partial_\sigma \Gamma^{\varepsilon}{}_{\mu\nu}$$

Cyclicly permute $\rho\nu\sigma$

$$\Rightarrow D_\rho R^\varepsilon{}_{\mu\nu\sigma} + D_\nu R^\varepsilon{}_{\mu\sigma\rho} + D_\sigma R^\varepsilon{}_{\mu\rho\nu} = 0$$

This is the Bianchi Identity.

Contract on ε & σ

$$\Rightarrow D_\rho R_{\mu\nu} - D_\nu R_{\mu\rho} + D_\varepsilon R^\varepsilon{}_{\mu\nu\rho} = 0$$

Contract on μ & ν

$$\Rightarrow D_\nu(2R^{\mu\nu} - g^{\mu\nu}R) = 0$$

A geometric Identity

This implies via Einstein Equation that

$$D_\mu T^{\mu\nu} = 0$$

Energy & Momentum is conserved!!

Eg Dust Blob

$$\begin{aligned} T^{\mu\nu} &= \rho(x)\tfrac{dx^\mu}{ds}\tfrac{dx^\nu}{ds} \\ &= \rho U^\mu U^\nu \end{aligned}$$

$$\begin{aligned} D_\nu T^{\mu\nu} &= D_\nu(\rho U^\mu U^\nu) \\ &= U^\mu D_\nu(\rho U^\nu) + \underbrace{(D_\nu U^\mu)\rho U^\nu}_{=0 \text{ via Geodesic Equ.}} = 0 \\ &= U_\mu(U^\mu D_\nu(\rho U^\nu) + (D_\nu U^\mu)\rho U^\nu)) \\ &= U_\mu U^\mu D_\nu(\rho U^\nu) + \underbrace{(D_\nu U^\mu)U_\mu}_{=0 \text{ since } U^\mu U_\mu = 1} \rho U^\nu \end{aligned}$$

$$\underline{\therefore D_\mu(\rho U^\nu) = 0}$$

Covariantized

$$\underline{\partial_\nu j^\nu = 0}$$

Summary

1. Reduces to Newtons Equation in weak limit
2. Both Sides have vanishing divergence & so corresponds to conservation of stress energy.
3. Can be derived from Variational Principle.
   $S = \int \sqrt{g} R \, d^4x$
   Einstein Hilbert Action

Compare EMF

Gauge Potential	Christoffel
A_μ	${\Gamma^\alpha}_{\mu\mu}$
Gauge Transformation	Coords. Transformation
$A_\mu \to +\partial_\mu \chi$	${\Gamma^\alpha}_{\mu\nu} \to \Lambda^{-1}\Lambda\Lambda\Gamma + \Lambda\frac{\partial^2 x}{\partial x \partial x}$
Field Strength	Curvature
$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= [D_\mu, D_\nu] \end{aligned}$	$\begin{aligned} {R^\alpha}_{\beta\mu\nu} &= \partial\Gamma - \partial\Gamma \\ &\quad + \Gamma\Gamma - \Gamma\Gamma \end{aligned}$
2-Maxwells Equ. (Identities)	Bianchi Identity
$D_{[\mu} F_{\nu\rho]} = 0$	$DR + DR + DR = 0$

Tests of GR

1. Advance of perihelion of Mercury known by Einstein
2. Deflection of star light by the sun Predicted
3. Gravitational Red Shift of spectral lines Predicted

Levemir (1845) measured the perihelion advancing by 5599" per century
but calculated 5599-43"
leaving 43"
Explanations: Unobserved planet, Vulcan

$$\frac{1}{r^2} \to \frac{1}{r^{2.00000016}}$$

General Relativity explain this.

Key Solution Schwarzschild

Solves $R_{\mu\nu} = 0$ for a spherically source
Why $R_{\mu\nu} = 0$; compare with EMF

$$\nabla^2 \phi = 0 \qquad \phi(r)$$

gives coulombs Pot!
If metric is spherically symmetric

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$R_{\mu\nu} = 0 \quad \Rightarrow \quad$ Differential Equation for $f(r)$ & $g(r)$
Solution:

$$\boxed{\begin{aligned} ds^2 &= -\left(1 - \tfrac{2GM}{r}\right) dt^2 + \left(1 - \tfrac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\ &= -\left(1 - \tfrac{2GM}{r}\right) dt^2 + \left(1 - \tfrac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2_{(2)} \end{aligned}}$$

This is the Schwarzschild solution.

It gives the metric around a Spherically Symmetric Source.

c.f. Coulombs Law $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}$

Examine Consequences:

Newtonian Approx. $\phi = -\frac{GM}{r}$

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2 + dy^2 + dz^2)$$

Planetary Motion

1. Approx: $r \to \infty \qquad g_{\mu\nu} \to \eta_{\mu\nu}$

$$2\frac{GM}{r_{\text{Sun}}} \sim 10^{-6} \qquad\qquad 2\frac{GM}{r_{\text{Earth}}} \sim 10^{-8}$$

$$\frac{ds}{dt} \sim 1 - 10^{-4}$$

   Newtonian Approx is Very Good.

2. Geodesics

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0$$

3. Solve Equ.: for r as a fn of θ, φ.

   the r equation, Goto Coords $u = \frac{1}{r}$

$$\frac{d^2u}{d\varphi^2} + u = \frac{M}{h^2} + 3Mu^2 \qquad\qquad h = r^2\frac{d\varphi}{ds}$$

c.f. Newtonian Eqn. $\frac{d^2u}{d\varphi^2} + u = \frac{M}{h^2}$

Newtonian Solution: $u = \frac{M}{h^2}(1 + e\cos(\varphi - \omega))$

G.R. Solution: $u = \frac{M}{h^2}(1 + e\cos(\varphi - (\omega + \frac{3M^2}{h^2})))$

(1st Approx)

$\delta\omega = \frac{3M^2}{h^2}$ Putting in G & π etc.

$$\delta\omega|_{1 \text{ rev}} = \frac{GM^2G^2\pi}{h^2}$$
$$\sim 42.9'' \text{ per century}$$

<u>Gravitational Deflection of Starlight by the Sun</u>

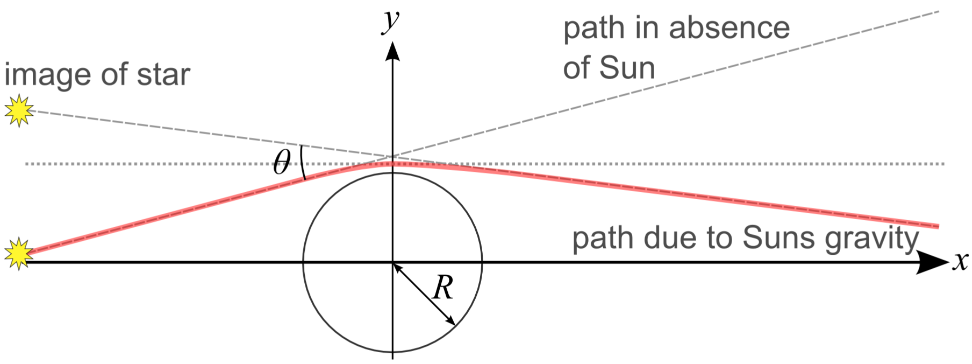

Figure 12: Path of a light ray is deflected due to the gravity of the Sun. θ is the angle of deflection.

In flat space: $ds^2 = -dt^2 + dr^2$

$$\therefore \frac{dr}{dt} = 1 \qquad \text{for } ds = 0$$

(units $c = 1$)

<u>with Sun</u>

$$\frac{dx}{dt} \sim +1\left(\frac{GM}{R}\right)$$
$$\frac{dy}{dt} \sim +0\left(\frac{GM}{R}\right)$$

We want deflection Angle, θ.

For small angles $\theta \sim \tan\theta$

$$\tan\theta = \frac{dy}{dx}|_{x=-\infty} - \frac{dy}{dx}|_{x=+\infty} = -\int_{-\infty}^{\infty}\left(\frac{d^2y}{dx^2}dx\right)$$

Metric: $ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)(dx^2 + dy^2 - rdz^2)$

In Newtonian Approx

Null Geodesics, parameterize path with λ.

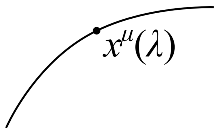

Figure 13:

$\frac{dx^\mu}{d\lambda} D_\mu \frac{dx^0}{d\lambda} = 0$ Geodesic

$\frac{d^2}{d\lambda^2} x^\mu + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$

Static Weak Field $\Rightarrow \Gamma^0{}_{\alpha\beta} = 0$

$\mu = 0$ equ. $\Rightarrow \frac{d^2 t}{d\lambda^2} = 0 \quad \therefore$ choose $t = \lambda$

$\frac{d^2 y}{dt^2} + \Gamma^y{}_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0$

But $\frac{dx}{dt} \sim 1 \quad \therefore \quad dt = dx$

Static $\Rightarrow \Gamma^y{}_{0x} = 0$

$$\therefore \frac{d^2 y}{dx^2} + \Gamma^y{}_{00} \frac{dt}{dt} \frac{dt}{dt} + \Gamma^y{}_{xx} \frac{dx}{dt} \frac{dx}{dt} = 0$$

$$\therefore \frac{d^2 y}{dx^2} + \Gamma^y{}_{00} + \Gamma^y{}_{xx} = 0$$

$$\begin{aligned} \Gamma^y{}_{00} &= -\tfrac{1}{2} \tfrac{\partial y_{00}}{\partial y} y^{yy} \Gamma^y{}_{xx} = -\tfrac{1}{2} \tfrac{\partial y_{xx}}{\partial y} y^{yy} \\ &= \tfrac{GM}{r^2} \tfrac{\partial r}{\partial y} + 0\left(\left(\tfrac{GM}{r}\right)^2\right) \\ &= \tfrac{GM_y}{(x^2+y^2)^{\frac{3}{2}}} \end{aligned}$$

$$ds^2 = -\left(1+2\phi\right)dt^2 + \left(1-2\phi\right)dr^2$$

Null for light $ds = 0$

$$\begin{aligned} \tfrac{dr}{dt} &= \sqrt{\tfrac{1+2\phi}{1-2\phi}} \\ &= \text{Effective vel. of light} \end{aligned}$$

Effective Medium with refractive index.

$\Rightarrow$ Lensing

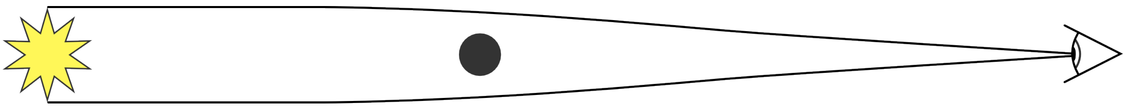

Figure 14:

$$\therefore \frac{d^2y}{dx^2} = -\frac{2GMy}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\therefore \theta = 2\int_{-\infty}^{\infty} \frac{GMy}{(x^2+y^2)^{\frac{3}{2}}} dx \sim \underbrace{2\int_{-\infty}^{\infty} \frac{GMR}{(x^2+y^2)^{\frac{3}{2}}} dx}_{y\sim R \text{ since } y\sim R+O\frac{GM}{R}} = 4\frac{GM}{R}$$

$$\text{For Sun:} \quad \begin{array}{rll} \theta & = & 1.75'' \quad \text{calculated} \\ & \sim & 1.7'' - 1.9'' \quad \text{observed} \end{array}$$

During Eclipse; first done by Eddington.

<u>Gravitational Red Shift</u>

Figure 15: photons from atom 2 are red shifted compared to atom 1.

Assume Atoms are stationary i.e. no ordinary Doppler shift.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

ds - interval between two events

Take $dr = 0$

$$\begin{array}{lrl} \text{On Star} & ds^2 = & g_{00}\left(r_2\right) dt_2^2 \\ \text{Earth} & = & g_{00}\left(r_1\right) dt_1^2 \end{array}$$

$$\delta t = \frac{1}{v}$$

Same Atoms $\Rightarrow ds_1 = ds_2$

(ds is proper time measured in natural coords.)

$$\therefore \frac{dt_2}{dt_1} = \sqrt{\frac{g_{00}(r_1)}{g_{00}(r_2)}} = \frac{v_1}{v_2}$$

Importantly though:

The interval between blips emitted at 2 is unchanged as they travel on a null geodesic to the eye.

Radial Geodesic $\frac{dt}{dr} = \frac{1}{1-\frac{2GM}{r}} dt = \int_{r_2}^{r_1} \frac{dr}{1-\frac{GM}{r}}$

Δt is the time between Emission & Reception & is time independent so two blips ΔT apart will arrive ΔT apart also.

Use Newtonian Approx

$$\begin{aligned} \frac{v_1}{v_2} &= \sqrt{\frac{1+2\phi(r_1)}{1+2\phi(r_2)}} \\ &\simeq \frac{1+2\phi(r_1)}{1+2\phi(r_2)} \sim 1 + \phi(r_1) - \phi(r_2) \end{aligned}$$

Earth & Sun:

$$\begin{aligned} r_2 &= R \\ r_1 &\sim \infty \end{aligned}$$

$$\phi(r_2) = -\frac{GM}{R} \qquad \phi(r_1) = 0$$

Note, in SI units we must reinstate c.

$\boxed{\frac{GM}{c^2 R}}$

Assume $\quad v_2 = v_1 + \delta v_1$

$$\frac{v_1}{v_1 + \delta v_1} = 1 - \frac{\delta v_1}{v_1}$$

$$\therefore \frac{\delta v_1}{v_1} = -\frac{GM}{R} < 0$$

$\Rightarrow$ Red Shift $\sim 10^-6$ for the Sun.

On Earth (1960): Over 70 ft!

$$\begin{aligned} \frac{\delta v}{v} &= 2.46 \times 10^{-15} \\ \text{Measured} \quad &= 2.77 \times 10^{-15} \pm 0.26 \times 10^{-15} \end{aligned}$$

Black Holes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

Blows up at $r = 0$ & $r = 2GM \equiv r_0$ "Schwarzschild radius".

Note, Beware Coord. Choice, cords do not always reflect geometry.

In fact, Geometry is perfectly smooth at $r = r_0$

i.e. Curvature is finite and can be small!

Not finite at $r = 0 \to \infty$

Important Point, Causal Structure is altered at $r = r_0$.

Flat Space: $ds^2 = -dt^2 + dx^2$

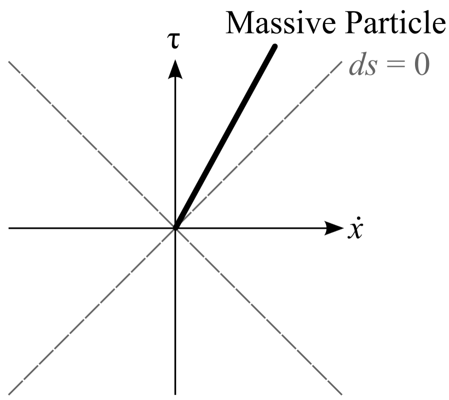

Figure 16:

Light has $ds = 0$

Massive Particle $ds < 0$

i.e. Time like trajectory

In curved space the light cones are only local defined by $ds = 0$.

So lets look at light cone structure of the Schwarzschild metric.

$$ds^2 = -\underbrace{\left(1 - \frac{r_0}{r}\right)}_{a} dt^2 + \underbrace{\left(1 - \frac{r_0}{r}\right)^{-1}}_{b} dr^2$$

Outside	$r > r_0$	$1 > a > 0$	$0 < b > 1$
Inside	$r < r_0$	$a < 0$	$b < 0$

$ds \leq 0$ for a future patch/region.

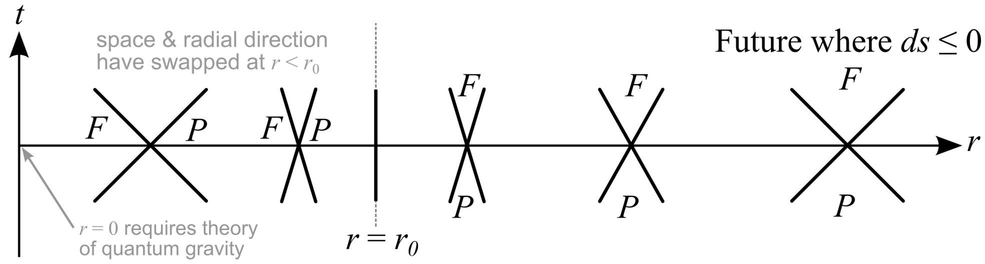

Figure 17:

For $ds = 0$

$\frac{dt}{dr} = \frac{1}{1 - \frac{r_0}{r}} = b$

"Time" is now directed toward $r = 0$.

The future is towards $r = 0$.

You cannot escape, the light cone rotates till your causal structure implies you must go to $r = 0$.

Falling into a black hole:

Frame of an infalling observer, radial motion only

$ds \leq 0$ for a future patch/region.

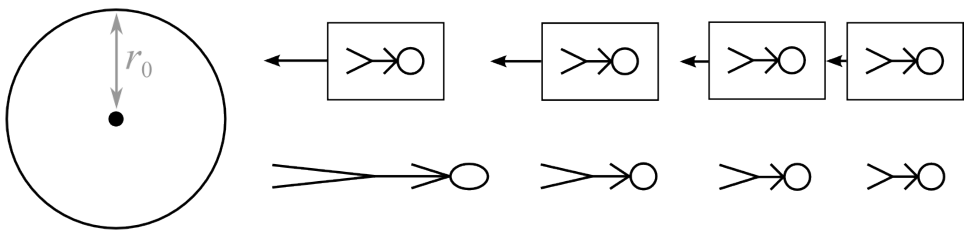

Figure 18:

$$ds^2 = -\gamma dt^2 + \gamma^{-1} dr^2 \qquad \gamma = 1 - \frac{r_0}{r}$$

Want $r(s)$

$$\frac{dt}{ds} = \frac{k}{\gamma} \quad \text{from geodesic Equations} k \text{is const. of integration}$$

Starts at ∞ with no velocity

$$r = \infty \qquad \gamma = 1$$

$$ds^2 = -dt^2 + dr^2$$

$$\left(\frac{ds}{dt}\right)\Bigg|_{r=\infty} = +i \qquad \Rightarrow \qquad k = i$$

(Note, $\quad \frac{ds}{dt}^2 = 1 - v^2 \quad$ in flat space at velocity v)

Relativists normally use $ds^2 = +dt^2 - dx^2$ so that $k = 1$.

$$\begin{aligned} \therefore ds^2 &= -\gamma dt^2 + \gamma^{-1} dr^2 \\ &= +\gamma \tfrac{ds^2}{\gamma^2} + \gamma^{-1} dr^2 \\ ds^2\left(-\gamma + 1\right) &= dr^2 \\ \therefore ds &= \tfrac{dr}{\sqrt{1-\gamma}} = \sqrt{\tfrac{r}{r_0}} dr \\ \therefore s &= s_0 - \tfrac{2}{3} r_0 \left(\tfrac{r}{r_0}\right)^{\frac{3}{2}} \end{aligned}$$

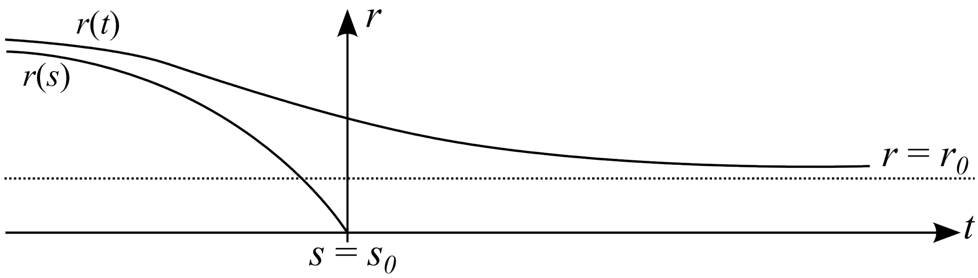

Figure 19:

S_0 is the time taken to fall to $r = 0$

$\therefore$ The proper time taken to fall from horizon to $r = 0$

$$= \frac{2}{3} r_0$$

From the frame of an observer watching at ∞, uses t as time

$$t = t_0 + r_0 \left(-\frac{2}{3}\left(\frac{r}{r_0}\right)^{\frac{3}{2}} - 2\left(\frac{r}{r_0}\right)^{\frac{1}{2}} + \log\left| \frac{\left(\frac{r}{r_0}\right)^{\frac{1}{2}} + 1}{\left(\frac{r}{r_0}\right)^{\frac{1}{2}} - 1} \right| \right)$$

$r(t)$ plotted above, the motion of an infalling observer seems frozen at the horizon. Also, ∞ gravitational red shift.

$$\frac{\overbrace{\lambda_d}^{\text{distance}}}{\underbrace{\lambda_e}_{\text{emitted}}} = \frac{\left(1 - \frac{r_0}{r_d}\right)^{\frac{1}{2}}}{\left(1 - \frac{r_0}{r_e}\right)^{\frac{1}{2}}}$$

More about black hole properties

Black holes have thermodynamic "properties".

Obey a first law

$$dE = T\ dS$$

2nd Law $\Delta S \geq 0$ in any process that is closed.

With $s = \frac{A}{4G_N}$ A - Area of the horizon.

T temperature, is the surface gravity at the horizon.

$s = \frac{A}{4G_N}$ is remarkable since this means the information inside a black hole dpends only on horizon area. The origin of the anthropic principle.

Black holes have a "non hair" theorem; that is the only information that the outside world can see is its mass, angular momentum & charge.

Remarkably, all once collapse begins all information is "lost" since there is only kerr(rotating) Schwarzschild black holes.

No matter what you begin with!

Cosmology

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$
$$H = 8\pi G \quad \text{useful}$$

Gravity dominates the Universe because there is no gravitational screening.

Approximations/Assumptions: Universe is Homogenous & Isotropic.

Time at big enough scales. ($\sim$ 10 MPc)

- Parsec $\sim$ 3.262 light years
- Galaxy Size $\sim$ 1 kPc
- Galaxy Separation $\sim$ 1 MPc
- Visible Universe $\sim$ 1 GPc

Parsec, Parallax Second

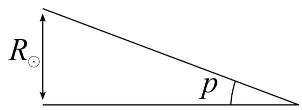

Figure 20: Parsec: Parallax Second. d-distince in parces $= 1/p$

d - distance in parsec $= \frac{1}{p}$

Hubble

From Galaxy red shifts,

$$z = \frac{\lambda - \lambda_e}{\lambda_e} = \underbrace{H}_{\text{Hubble const.}} \underbrace{D}_{\text{distance}}$$

$$\underbrace{v}_{\text{velocity of galaxies}} = HD \qquad \text{Expansion of the Universe}$$

$\frac{1}{H}$ is Hubble time, the upper limit on the age of the Universe

$$\frac{1}{H} \sim 20 \times 10^9 \text{ years}$$

Xenophanes of Colophon - God is an eternal sphere.

Parmenides - God is an infinite expanding sphere.

Alain de Lille (12c) God is an intelligible sphere whose centre is everywhere & circumference nowhere.

Pascal Nature is an infinite sphere whose centre is everywhere & whose circumference is nowhere.

* As in original text “effroyable”.

To see this expansion consider dots on the surface of a balloon.

Blow the balloon up and see how the dots move away from each other.

This is expansion with $\underline{\underline{\text{no}}}$ centre.

Literacy fun, "The fearful sphere of Pascal", Jorge Luis Borges.

Cosmological Principle

The Uinverse is Homogeneous;

The Universe is Isotropic

As viewed by an observer at rest WRT motion of his neighbourhood ie. "comoving observer".

Treat Galaxies as a sort of gas/fluid.

$$T^{\mu\nu} = (\rho + \eta)U^\mu U^\nu - g^{\mu\nu}\eta$$
$$U^\mu = \frac{dx^\mu}{ds}$$

The Universe is evolving in time; the cosmological principle can be thought of as saying that the history of the Universe is independent of the observer ie. each comoving observer sees the Universe & its history to be the same.

Cosmic time is the proper time for a comoving observer.

Take a metric which has proper time = time & is isotropic.

$$\Rightarrow ds^2 = -dt^2 + g_{ij}dx^i dx^j$$

$$d\Omega_{(2)}^2 = dt^2 + \sin^2\theta \; d\phi^2$$

$$\left.\begin{array}{lcc} 1. & \text{Flat spaces} & R=0 \\ 2. & \text{Sphere} & R>0 \\ 3. & \text{Coonst -ve curvature} & R<0 \end{array}\right\} \quad \text{3-curvature} \quad \left\{\begin{array}{l} \text{"}\mathbb{R}^3\text{"} \\ \text{"}\mathbb{S}^3\text{"} \\ \text{"}\mathbb{H}^3\text{"} \end{array}\right.$$

Metric is:

$$\begin{array}{rcll} ds^2_{(3)} & = & dr^2 + r^2\ d\Omega^2_{(2)} & \text{Flat } \mathbb{R}^3 \\ & = & \frac{R^2}{R^2-r^2} dr + r^2\ d\Omega^2_{(2)} & \mathbb{S}^3 \\ & = & \frac{R^2}{R^2+r^2} dr + r^2\ d\Omega^2_{(2)} & \mathbb{H}^3 \end{array}$$

$$\sigma \equiv \frac{r}{R}$$

$$ds^2 = -dt^2 + R^2(t)\left(\frac{d\sigma^2}{1-\kappa\sigma^2} + \sigma^2\ d\Omega^2_{(2)}\right)$$

"Robertson Walker Metric"

$$\begin{array}{rcll} \kappa & = & 0 & \text{Flat} \\ \kappa & = & 1 & \text{3-Sphere} \\ \kappa & = & -1 & \mathbb{H}^3 \end{array}$$

There only remains one function, $R(t)$.

Proper time = Cosmological time

t, σ, θ, φ are natural "freely falling"/"comoving" coords.

Unless acted on by some exterior force then we must be at rest with respect to them.

Therefore a galaxy will have the same t, σ, θ, φ for all time

Expansion of the Universe

$$R(t) \text{ with } \dot{R}(t) > 0$$

Distances between galaxies (same σ & φ) $\sigma' = 0$ & $\sigma' = \sigma$

$$\begin{array}{rcl} D(t) & = & \int dl = R(t) \int_0^\sigma \frac{d\sigma'}{\sqrt{1-\kappa\sigma'^2}} \\ \dot{D}(t) & = & \dot{R}(t) \frac{d\sigma'}{\sqrt{1-\kappa\sigma'^2}} \\ & = & \frac{\dot{R}(t)}{R} D(t) \end{array}$$

$$\dot{D}(t) \equiv HD$$

$$\text{with } H = \frac{\dot{R}(t)}{R(t)}$$

Work out $R(t)$ from Einsteins Equation's.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$8\pi G \rho = -\lambda + 3 \left(\frac{\dot{R}}{R} \right)^2 + 3 \frac{k}{R^2}$$

$$8\pi G p = \lambda - \left(\frac{\dot{R}}{R} \right)^2 - 2 \frac{\ddot{R}}{R} - \frac{k}{k^2}$$

Use Equ. of state $p = \eta(\rho)$

Case $\lambda = 0$

$$dE + \eta \; dV = 0$$

$$d(R^3 \rho) + \eta \; dR^3 = 0$$

What about Cosmological Constant?

Back to Time Component of Einstein Eqn.

$$3\left(\frac{\dot{R}}{R}\right)^2 - \lambda = 0$$

$$\Rightarrow \dot{R}^2 - \frac{\lambda}{3}R^2 = 0$$

$$\Rightarrow \dot{R} - \sqrt{\frac{\lambda}{3}} = 0$$

$$\Rightarrow R = e^{H(t-t_0)} \qquad\qquad H = \sqrt{\frac{\lambda}{3}}$$

Matter Dominated: $\Rightarrow \eta = 0$ "Friedmann Universe"

$$\begin{aligned} \Rightarrow \quad R^3\rho &= \text{constant} \\ &= \rho_0 R_0^3 \end{aligned}$$

$$\therefore \rho = \frac{\rho_0 R_0^3}{R^3}$$

Eliminate ρ

$$\Rightarrow \quad \dot{R}^2 = -\kappa + \frac{c}{R}$$

$\kappa = 0$ solution

$$\begin{aligned} \dot{R}^2 &= \tfrac{c}{R} \\ \therefore \quad R^{1/2}\dot{R} &= c^{1/2} \\ \therefore \quad \tfrac{2}{3}R^{3/2} &= c^{1/2}t \\ \therefore \quad R(t) &= \left(\tfrac{3}{2}\right)^{2/3} c^{1/3} t^{2/3} \end{aligned}$$

Radiation dominated: $\Rightarrow \eta = \frac{1}{3}\rho$ photon gas

= Eliminate & between the two independent Einstein Equations.

$$+\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{\kappa}{R^2} = 0$$

$\kappa = 0$ solution

$$+\ddot{R} + \left(\frac{\dot{R}}{R}\right)^2 = 0$$

$$\Rightarrow \qquad R(t) = at^{1/2}$$

λ dominated:

$$R(t) = e^{H(t-t_0)}$$

Our Universe begins	Radiation dominated
then	Matter dominated
then	λ dominated

Kaluza - Klein Theory

If space can bend & curve why not bend round & rejoin itself ie. a circle (we have already seen spherical spaces)

If space can be of finite size then this size can be in principle very small eg 10^{-30} m.

So why limit ourselves to four dimensions of space & time why not consider eg five dimensions with one dimension a circle of radius R (& R very small).

Consider then writing our five dimensional metric as (with cords (x^μ, θ)):

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \left(\begin{array}{c|c} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ \hline A_\nu & 1 \end{array}\right)$$

That is

$$g_{\hat{\mu}\nu} = A_\nu$$

$$g_{55} = 1 \qquad\qquad \oint_{2\pi R}^{0} \sqrt{g_{55}} = 2\pi R$$

Stick this Five Dimensional Metric into Einstein equations & require:

$$\partial_5 = 0$$

So that there is only dependence on x not on θ.

Calculating the Einstein Equations for this 5 dim metric gives the following.

Introduce $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

μ, ν components of Einstein are:

$$R^{(4)}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G^{(4)} T_{\mu\nu} \tag{1}$$

$$T_{\mu\nu} = -\frac{1}{4\pi}\left(F_{\mu\alpha}F^{\alpha}{}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}F^{\beta\alpha}\right)$$

$$G^{(4)} = \frac{G^{(5)}}{R}$$

$\mu, 5$ components:

$$D^{(4)}_{\nu} F^{\mu\nu} = 0 \tag{2}$$

$5, 5$ component does not give anything.

We recognise (2) as Maxwells Equations for the field strength

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

See Assignment Sheet.

$T_{\mu\nu}$ = Stress Energy of the Electromagnetic field.

<u>eg</u> $T_{00} = E^2 + B^2$

$g_{\mu 5} = A_\mu$ is the vector pot!

$$A_\mu = \begin{pmatrix} \phi \\ A_i \end{pmatrix} \begin{matrix} \leftarrow \text{Electric Potential} \\ \leftarrow \text{Magnetic Pot!} \end{matrix}$$

(Note, $A_\mu \to A_\mu + \partial_\mu \chi$ leaves $F_{\mu\nu}$ invariant in coord transformation)

If we are careful we find that

$$R^{-1} \sim \text{ Electric Charge, } e$$

We assumed that $\partial_\theta = 0$

Now consider a partical moving around the circle.

The momentum is quantized

$$\therefore p_\theta \sim \frac{n}{R} \sim ne$$
n is an integer.

Therefore the charge of any particle is quantized & is given by the momentum around the 5-th dimension.

Problems

The radius should also be dynamical not fixed; this gives rise to a new field that we dont see.

What about other forces?

More compleciated & higher dimensional geometry.

eg $\mathbb{S}^3$ gives something that looks like the weak Nuclear force.

Do we ever really need anything else?

Glossary

SR (S.R.) = Special Relativity

GR (G.R.) = General Relativity

Inertial Frame = a frame without acceleration.

Screening = The effect of a field being canceled out or screened by oposite charges.

c.f.

$$\rho_M = \text{mass density}$$

$$\rho_Q = \text{charge density}$$

$$l_p = \text{Planck Length} \approx 1.6 \times 10^{-35}$$

Greek letters for 4-D vectors (e.g. (t, x, y, z))

Latin letters for 3-D spacial vectors (e.g. (x, y, z))

$$\begin{pmatrix} t \\ x^i \end{pmatrix} = \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\frac{\partial}{\partial x^\nu} \equiv \underbrace{\partial_\nu}_{\substack{\text{short} \\ \text{form}}}$$

$$\text{Cross/Vector-Product} = \times = \wedge$$