RELATIVITY & GRAVITATION

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SECTION 3 - GRAVITATIONAL WAVES AND PERTURBATION THEORY

3.1 Perturbation theory

Perturbation theory is a widely used tool for considering physical scenarios that are more complicated than a single body. The idea is to determine which parts of the geometry can be neglected, when gravity is weak, and then systematically ignoring them. Our starting point is

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}}$$

where $|h_{\mu\nu}| \ll 1$, which means we will be considering spacetimes that are close to Minkowski. This approximation is sufficient to describe everything in the Solar System, and even the gravitational field of black holes (as long as we stay far away from their horizons).

Exercise: show that the condition $g^{\mu\nu}g_{\nu\sigma} = \delta^\mu{}_\sigma$ means that

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2)$$

where $h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$, and where $O(h^2)$ indicates (small) terms that are of size h^2 (or smaller)

solution: if $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2)$, then we should be able to write $g^{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu} + O(h^2)$, where $f^{\mu\nu} \sim h$.

$$\Rightarrow \delta^\mu{}_\sigma = g^{\mu\nu}g_{\nu\sigma} = (\eta^{\mu\nu} + f^{\mu\nu})(\eta_{\nu\sigma} + h_{\nu\sigma}) + O(h^2)$$

$$= \eta^{\mu\nu}\eta_{\nu\sigma} + \eta^{\mu\nu}h_{\mu\nu} + f^{\mu\nu}\eta_{\nu\sigma} + O(h^2)$$

$$= \delta^\mu{}_\sigma + \eta^{\mu\nu}h_{\nu\sigma} + f^{\mu\nu}\eta_{\nu\sigma} + O(h^2)$$

$$\Rightarrow f^{\mu\nu}\eta_{\nu\sigma} = -\eta^{\mu\nu}h_{\nu\sigma}$$

Now multiply through by $\eta^{\sigma\rho}$:

$$\Rightarrow f^{\mu\nu}\eta_{\nu\sigma}\eta^{\sigma\rho} = f^{\mu\nu}\delta_\nu{}^\rho = f^{\mu\rho} = -\eta^{\mu\nu}\eta^{\sigma\rho}h_{\nu\sigma}$$

$$\Rightarrow g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

where $h^{\mu\nu} = \eta^{\mu\sigma}\eta^{\nu\rho}h_{\sigma\rho}$.

Substituting these expressions into the definition of the Christoffel symbols gives

$$\boxed{\Gamma^{\sigma}{}_{\mu\nu} = \Gamma^{(1)\sigma}{}_{\mu\nu} + \Gamma^{(2)\sigma}{}_{\mu\nu} + O(h^3)}$$

where numbers in brackets indicate order-of-smallness in h. It is straightforward to find

$$\Gamma^{(1)\sigma}{}_{\mu\nu} = \frac{1}{2}\eta^{\sigma\rho}(\partial_{nu}h_{\rho\mu} + \partial_{\mu}h_{\rho\nu} - \partial_{\rho}h_{\mu\nu})$$

and

$$\Gamma^{(2)\sigma}{}_{\mu\nu} = -\frac{1}{2}h^{\sigma\tau}(\partial_{\nu}h_{\tau\mu} + \partial_{\mu}h_{\tau\nu} - \partial_{\tau}h_{\mu\nu})$$

Similarly, these can be substituted into the definition of $R_{\mu\nu}$ to find

$$\boxed{R_{\mu\nu} = R^{(1)}{}_{\mu\nu} + R^{(2)}{}_{\mu\nu} + O(h^3)}$$

where

$$R^{(1)}{}_{\mu\nu} = \frac{1}{2}(\partial_{\mu}\partial_{\rho}h^{\rho}{}_{\mu} + \partial_{\rho}\partial_{\mu}h^{\rho}{}_{\nu} - \partial_{\mu}\partial_{\nu}h^{\rho}{}_{\rho} - \Box h_{\mu\nu})$$

and

$$
\begin{aligned}
R^{(2)}{}_{\mu\nu} = \frac{1}{2}h^{\rho\sigma}(\partial_\mu\partial_\nu h_{\rho\sigma} + \partial_\rho\partial_\sigma h_{\mu\nu} - \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\nu\partial_\sigma h_{\nu\rho}) \\
+\frac{1}{2}(\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h^\sigma{}_\sigma)(\partial_\rho h_{\mu\nu} - \partial_\mu h_{\nu\rho} - \partial_\nu h_{\mu\rho}) \\
+\frac{1}{4}\partial_\mu h^{\rho\sigma}\partial_\nu h_{\rho\sigma} + \frac{1}{2}\partial^\sigma h^\rho{}_\nu(\partial_\sigma h_{\rho\mu} - \partial_\rho h_{\sigma\mu})
\end{aligned}
$$

The indices in these last equations have been raised using $\eta^{\mu\nu}$ (e.g. $h^\rho{}_\mu \equiv \eta^{\rho\nu}h_{\nu\mu}$). We have also used $\Box \equiv \partial^\mu\partial_\mu$.

We can now use these expressions to get perturbative approximations to the field equations. For example, to leading order

$$
R^{(1)} = \eta^{\mu\nu}R^{(1)}{}_{\mu\nu} = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h^\mu{}_\mu
$$

$$
\Rightarrow G^{(1)}{}_{\mu\nu} = R^{(1)}{}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} = 8\pi G T_{\mu\nu} + O(h^2)
$$

$$
\Rightarrow \boxed{\partial_\nu\partial_\nu h^\rho{}_\mu + \partial_\rho\partial_\mu h^\rho{}_\nu - \partial_\mu\partial_\nu h^\rho{}_\rho - \Box h_{\mu\nu} + \eta_{\mu\nu}(\Box h^\rho{}_\rho - \partial_\rho\partial_\sigma h^{\rho\sigma}) = 16\pi G T_{\mu\nu} + O(h^2)}
$$

This equation is the one we will now try and solve. It can be simplified by defining

$$
\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\rho{}_\rho
$$

$$
\Rightarrow \boxed{\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma} - \partial_\nu\partial_\rho\bar{h}^\rho{}_\nu = -16\pi G T_{\mu\nu} + O(h^2)}
$$

Example: show that $\bar{h}_{\mu\nu}$ is the trace-reverse of $h_{\mu\nu}$, so that

$$\bar{h}^{\rho}{}_{\rho} = -h^{\rho}{}_{\rho} \qquad \text{and} \qquad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}^{\rho}{}_{\rho}\,.$$

Solution: if $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\rho}{}_{\rho}$ then

$$\begin{aligned}
\bar{h}_{\rho}{}^{\rho} &= \bar{h}_{\mu\nu}\eta^{\mu\nu} \\
&= h_{\mu\nu}\eta^{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\mu\nu}h^{\rho}{}_{\rho} \\
&= h_{\rho}{}^{\rho} - \frac{1}{2}\delta_{\mu}{}^{\mu}h_{\rho}{}^{\rho} \\
&= h_{\rho}{}^{\rho} - \frac{1}{2}\times 4 \times h_{\rho}{}^{\rho} \\
&= h_{\rho}{}^{\rho} - 2h_{\rho}{}^{\rho} \\
&= -h_{\rho}{}^{\rho}
\end{aligned}$$

and

$$\begin{aligned}
\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}^{\rho}{}_{\rho} &= h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h_{\rho}{}^{\rho} - \frac{1}{2}\eta_{\mu\nu}(-h_{\rho}{}^{\rho}) \\
&= h_{\mu\nu}
\end{aligned}$$

So the proposed equations are true.

3.2 Gauge transformations

The general covariance of Einstein's equations is broken when we start approximating them using perturbation theory. What is left is covariance under "small" coordinate transformations:

$$x'^{\mu} = x^{\mu} + \xi^{\mu}$$

where $|\xi^{\mu}| \sim |h_{\mu\nu}|$ (where $\sim$ means they are approximately the same size and ξ^{μ} is a quantity known as a gauge generator). These are called "gauge transformations", and the coordinate freedom that existed in the full theory is replaced by the "gauge freedom" to choose ξ^{μ}

Differentiating this gives the gauge transformation matrix

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}{}_{\nu} + \partial_{\nu}\xi^{\mu}$$

Requiring $\frac{\partial x'^{\mu}}{\partial x^{\nu}}\frac{\partial x^{\nu}}{\partial x'^{\rho}} = \delta^{\mu}{}_{\rho}$ then gives the inverse:

$$\frac{\partial x^{\mu}}{\partial x'^{\nu}} = \delta^{\mu}{}_{\nu} - \partial_{\nu}\xi^{\mu} + O(h^2)$$

These correspond to infinitesimal coordinate transformations.

Let us now perform a gauge transformation on the metric, using the infinitesimal coordinate transformation above:

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma} = (\delta^\rho{}_\mu - \partial_\mu \xi^\rho)(\delta^\sigma{}_\nu - \partial_\nu \xi^\sigma)(\eta_{\rho\sigma} + h_{\rho\sigma})$$

$$= \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + O(h^2)$$

where we have written $\xi_\mu = \eta_{\mu\nu}\xi^\nu$. This can be compared to the perturbative expression for $g'_{\mu\nu}$, in the new coordinate system:

$$g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu}$$

$$\Rightarrow \boxed{h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + O(h^2)}$$

This equation gives the effect that a gauge transformation has on metric perturbations. It is extremely useful for simplifying calculations in perturbation theory.

3.3 The Lorenz gauge

The trace-reversed metric perturbation gauge transforms as follows:

$$\bar{h}'^{\mu\nu} = \bar{h}^{\mu\nu} - \partial^\mu \xi^\nu - \partial^\nu \xi^\mu + \eta^{\mu\nu} \partial_\rho \xi^\rho$$

This means that

$$\partial_\nu \bar{h}'^{\mu\nu} = \partial_{\mu\nu} \bar{h}^{\mu\nu} - \Box \xi^\mu$$

If we now choose $\Box \xi^\mu = \partial_\nu \bar{h}^{\mu\nu}$ then

$$\boxed{\partial_\nu \bar{h}'^{\mu\nu} = 0}$$

This choice of ξ^μ results in the "Lorenz gauge". It's extremely useful because it simplifies the leading-order perturbed field equations to

$$\boxed{\Box \bar{h}'_{\mu\nu} = -16\pi G T_{\mu\nu}}$$

Note: the Lorentz gauge is preserved by any addition gauge transformation, provided that ξ^μ satisfies $\Box \xi^\mu = 0$.

Exercise: prove that $\bar{h}^{\mu\nu}$ transforms as

$$\bar{h}'^{\mu\nu} = \bar{h}^{\mu\nu} - \partial^\mu \xi^\nu - \partial^\nu \xi^\mu + \eta^{\mu\nu} \partial_\rho \xi^\rho$$

Solution: under the transformation $x'^\mu = x^\mu + \xi^\mu$ we have

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + O(h^2)$$

now

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h'_\rho{}^\rho = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu - \frac{1}{2}\eta_{\mu\nu}(h_\rho{}^\rho - 2\partial_\rho \xi^\rho)$$

$$= \bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu} \partial_\rho \xi^\rho$$

Raising indices with $\eta^{\mu\sigma}$ and $\eta^{\nu\rho}$ then gives the required result.

<u>Example</u>: we can linearize the Schwarzschild solution:

$$ds^2 = -\left(1 - \frac{2Gm}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2Gm}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,.$$

If we assume $\frac{2Gm}{r} \ll 1$ then

$$\begin{aligned} ds^2 &\simeq -\left(1 - \frac{2Gm}{r}\right) dt^2 + \left(1 + \frac{2Gm}{r}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + O\left(h^2\right) \\ &= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{2Gm}{r}(dt^2 + dr^2) + O\left(h^2\right) \\ &= \eta_{\mu\nu} dx^\mu dx^\nu + h_{\mu\nu} dx^\mu dx^\nu + O(h^2)\,, \end{aligned}$$

so that

$$\Rightarrow \quad h_{\mu\nu} dx^\mu dx^\nu = \frac{2Gm}{r}(dt^2 + dr^2)\,.$$

This perturbation is traceless ($\eta^{\mu\nu} h_{\mu\nu} = 0$), so $h_{\mu\nu} = \bar{h}_{\mu\nu}$.

Note: this perturbation is <u>not</u> in Lorenz gauge, as

$$\partial_t \bar{h}^{tt} = \partial_t \left(\frac{2Gm}{r}\right) = 0$$

but

$$\partial_r \bar{h}^{rr} = \partial_r \left(\frac{2Gm}{r}\right) = -\frac{2Gm}{r^2} \neq 0\,.$$

3.4 Linearised solutions in vacuum

In vacuum $T_{\mu\nu} = 0$, and the field equations in the Lorenz gauge reduce to

$$\Box \bar{h}_{\mu\nu} = 0$$

with

$$\partial_\mu \bar{h}^{\mu\nu} = 0$$

A solution to these equations can be written as

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \exp(ik_\rho x^\rho)$$

where $A_{\mu\nu}$ is a symmetric matrix of complex, constant values, and where k_μ are the constant, real components of a vector. We must take the real part of $\bar{h}_{\mu\nu}$ if we want a real spacetime, of course.

Exercise: show that the expression for $\bar{h}_{\mu\nu}$ is a solution to $\Box \bar{h}_{\mu\nu} = 0$ if

$$k^\mu k_\mu = 0$$

and that it satisfies the gauge condition if

$$A^{\mu\nu} k_\nu = 0$$

Solution: if $\bar{h}_{\mu\nu} = A_{\mu\nu} \exp(ik_\rho x^\rho)$ then

$$\begin{aligned}
\Box \bar{h}_{\mu\nu} &= \eta^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} A_{\mu\nu} \exp(ik_\rho x^\rho) \\
&= \eta^{\alpha\beta} ik_\alpha k_\beta A_{\mu\nu} \exp(ik_\rho x^\rho) \\
&= -k^\alpha k_\alpha \bar{h}_{\mu\nu} \\
&= 0 \qquad \text{if} \qquad k^\mu k_\mu = 0
\end{aligned}$$

and

$$\begin{aligned}
\partial_\mu \bar{h}^{\mu\nu} &= \partial_\mu \left(A^{\mu\nu} \exp(ik_\rho x^\rho) \right) \\
&= ik_\mu A^{\mu\nu} \exp(ik_\rho x^\rho) \\
&= 0 \qquad \text{if} \qquad k_\mu A^{\mu\nu} = 0\,.
\end{aligned}$$

The linearised Einstein equations and Lorenz gauge condition are therefore satisfied if $k^\mu k_\mu = 0$ and $k_\mu A^{\mu\nu} = 0$.

3.5 Transverse-traceless gauge

To investigate gravitational waves it is often useful to specialise to the "transverse-traceless" gauge. Recall that in Lorenz gauge there existed the residual gauge freedom

$$x'^{\mu} = x^{\mu} + \xi^{\mu}$$

where $\Box\xi^{\mu} = 0$. This freedom can be used to enforce the additional conditions

$$\boxed{\bar{h}^{\mathrm{TT}}_{0i} = 0 \qquad \text{and} \qquad \bar{h}^{\mathrm{TT}\mu}{}_{\mu} = 0}$$

as well as the usual Lorenz condition $\partial_{\mu}\bar{h}^{\mu\nu} = 0$, which now becomes

$$\boxed{\partial_t \bar{h}^{00}_{\mathrm{TT}} = 0 \qquad \text{and} \qquad \partial_i \bar{h}^{ij}_{\mathrm{TT}} = 0}$$

because of the first of the conditions above. This uses up all of our gauge freedoms. If we now consider the linearised solution in vacuum, $\bar{h}_{\mu\nu} = A_{\mu\nu}\exp(ik_{\rho}x^{\rho})$, then these four equations imply

$$A^{0i}_{\mathrm{TT}} = A_{\mathrm{TT}}{}^{\mu}{}_{\mu} = A^{00}_{\mathrm{TT}} = A^{ij}_{\mathrm{TT}}k_j = 0\,.$$

If these conditions are satisfied then we are in TT gauge.

As an example, let's take k^μ to point in the z-direction

$$k^\mu = (\omega, 0, 0, k)$$

The conditions $k^\mu k_\mu = 0$ and $A_{\mu\nu}k^\nu = 0$ then imply $\omega = -k$ and $A_{\mu 3} = A_{\mu 0}$. This gives

$$A_{\mu\nu} = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{01} & A_{11} & A_{12} & A_{10} \\ A_{02} & A_{12} & A_{22} & A_{02} \\ A_{00} & A_{01} & A_{02} & A_{00} \end{pmatrix}$$

A gauge generator that satisfies $\Box\xi^\mu = 0$ is then given by

$$\xi^\mu = \varepsilon^\mu \exp(ik_\rho x^\rho)$$

where ε^μ is a constant vector, and k^μ is the same vector as above. The components of this transformation are

$$A'_{00} = A_{00} - ik(\varepsilon_0 + \varepsilon_3), \qquad A'_{12} = A_{12}$$

$$A'_{11} = A_{11} - ik(\varepsilon_0 - \varepsilon_3), \qquad A'_{01} = A_{01} - ik\varepsilon_1$$

$$A'_{22} = A_{22} - ik(\varepsilon_0 - \varepsilon_3), \qquad A'_{02} = A_{02} - ik\varepsilon_2$$

If we therefore make the choices

$$\varepsilon_0 = -\frac{i}{4k}(2A_{00} + A_{11} + A_{22}), \qquad \varepsilon_1 = -\frac{i}{k}A_{01}$$

$$\varepsilon_2 - \frac{i}{k}A_{02}, \qquad \varepsilon_3 = -\frac{i}{4k}(2A_{00} - A_{11} - A_{22})$$

then we have $A'_{00} = A'_{01} = A'_{02} = 0$, and

$$A'_{11} = -A'_{22} = \frac{1}{2}A_{11} - \frac{1}{2}A_{22}$$

This gives

$$\Rightarrow \qquad A'_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A'_{11} & A'_{12} & 0 \\ 0 & A'_{12} & -A'_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the perturbation to the metric is then

$$\bar{h}^{TT}_{\mu\nu} = h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_x & 0 \\ 0 & h_x & h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $h_+ \equiv A_{11}\exp(ik_\rho x^\rho)$ and $h_\times \equiv A_{12}\exp(ik_\rho x^\rho)$.

Another (often simpler) way to put $A_{\mu\nu}$ into the TT gauge is to introduce the "projection tensor"

$$P_{ij} = \delta_{ij} - n_i n_j$$

where n_i obeys $n^i n_i = 1$ and points in the direction of propagation of the gravitational wave. We will use this result in Section 3.12 to calculate the energy that gravitational waves remove from binary systems.

Example: show that the following corresponds to a perturbation in TT gauge:

$$A^{ij}_{\rm TT} = \left(P^i_{\ k} P^j_{\ l} - \frac{1}{2} P^{ij} P_{kl} \right) A^{kl}$$

where $P_{ij} = \delta_{ij} - n_i n_j$ is the projection tensor, and n_i is a space-like unit vector that points in the direction of propagation of the gravitational wave.

Solution: we still want $\bar{h}^{\mu\nu} = A^{\mu\nu}_{\rm TT} e^{ik_\rho x^\rho}$ to satisfy Einstein's equations and the Lorenz gauge condition, so require $k_\mu A^{\mu\nu}_{\rm TT} = 0$.

$$\Rightarrow \quad A^{0\mu}_{\rm TT} k_\mu = 0 \qquad \text{where} \qquad k^\mu = (\omega, \omega n_i)$$

$$\Rightarrow \quad -A^{00}_{\rm TT} k_0 + A^{0i}_{\rm TT} \omega n_i = 0$$

$$\Rightarrow \quad A^{00}_{\rm TT} = A^{0i}_{\rm TT} n_i \tag{1}$$

and

$$A^{i\mu}_{\rm TT} k_\mu = 0$$

$$\Rightarrow \quad -A^{i0}_{\rm TT}\omega + A^{ij}_{\rm TT}\omega n_j = 0$$

$$\Rightarrow \quad A^{i0}_{\rm TT} = A^{ij}_{\rm TT} n_j \tag{2}$$

Equations (1) and (2) together give

$$A^{00}_{\rm TT} = A^{ij}_{\rm TT} n_i n_j \,.$$

Now calculate $P^i{}_j n_i = (\delta^i{}_j - n^i n_j) n_i = n_j - n_j = 0$, which gives

$$\Rightarrow \quad A^{ij}_{\rm TT} n_j = 0 \qquad \text{and} \qquad A^{ij}_{\rm TT} n_i n_j = 0$$

so $$\boxed{A^{i0}_{\rm TT} = 0} \qquad \text{and} \qquad \boxed{A^{00}_{\rm TT} = 0}.$$

Next we need to evaluate ${A_{\rm TT}}^\mu{}_\mu$ and $A^{ij}_{\rm TT} k_j$.

If

$$A^{00}_{\rm TT} = 0 \qquad \text{then} \qquad {A_{\rm TT}}^\mu{}_\mu = {A_{\rm TT}}^i{}_i,$$

but

$${A_{\rm TT}}^i{}_i = (P^i{}_k P_{il} - \frac{1}{2} P^i{}_i P_{kl}) A^{kl},$$

where

$$
\begin{aligned}
P^i{}_k P_{il} &= (\delta^i{}_k - n^i n_k)(\delta_{il} - n_i n_l) \\
&= \delta_{lk} - n_k n_l - n_l n_k + n_k n_l \\
&= \delta_{lk} - n_k n_l \\
&= P_{lk}
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{1}{2} P^i{}_i P_{kl} &= -\frac{1}{2}(\delta^i{}_i - n^i n_i) P_{kl} \\
&= -\frac{1}{2}(3-1) P_{kl} \\
&= -P_{kl}
\end{aligned}
$$

so ${A_{\mathrm{TT}}}^i{}_i = 0$, which implies $\boxed{{A_{\mathrm{TT}}}^\mu{}_\mu} = 0$. Finally,

$$
\begin{aligned}
A^{ij}_{\mathrm{TT}} k_j &= (P^i{}_k P^j{}_l - \frac{1}{2} P^{ij} P_{kl}) A^{kl} k_j \\
&= \omega (P^i{}_k P^j{}_l - \frac{1}{2} P^{ij} P_{kl}) A^{kl} n_j
\end{aligned}
$$

$$
\Rightarrow \quad \boxed{A^{ij}_{\mathrm{TT}} k_j = 0} \qquad \text{as} \qquad P^{ij} n_j = 0\,.
$$

The four boxed equations are the conditions to be in TT gauge, so the proposition is true.

3.6 The effect of gravitational waves

Let's keep working in TT gauge, and consider the effect of our plane gravitational wave on a test particle with 4-velocity $u^\mu = (1,0,0,0)$. The geodesic equation then gives

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu{}_{\rho\sigma} u^\rho u^\sigma = -\frac{1}{2}\eta^{\mu\nu}(\partial_t h_{\mu t} + \partial_t h_{t\mu} - \partial_\nu h_{tt}) + O(h^2)$$

Recall that in TT gauge $h^{\rm TT}_{\nu t} = 0$ for all time. This means

$$\frac{du^\mu}{d\tau} = 0$$

i.e. the particle stays at fixed spatial coordinates as the gravitational wave passes though. This is an important result, but it does not mean that the gravitational wave has no effect (as we will now see).

Consider two test particles, separated by coordinate distance Δx in the x-direction. The proper distance between them is therefore

$$L_x \equiv \int_0^{\Delta x} \sqrt{g_{xx} dx dx} = \int_0^{\Delta x} \sqrt{\eta_{xx} + h_{xx}} dx = \int_0^{\Delta x} \left(1 + \frac{1}{2} h_{xx}\right) dx + O(h^2)$$

$$= \int_0^{\Delta x} \left(1 + \frac{A_{11}}{2}\cos(kz - \omega t)\right) = \left(1 + \frac{A_{11}}{2}\cos(kz - \omega t)\right)\Delta x$$

Likewise, two particles separated by coordinate distance Δy in the y-direction have a proper distance between them of

$$L_y = \left(1 - \frac{A_{22}}{2}\cos(kz - \omega t)\right)\Delta y$$

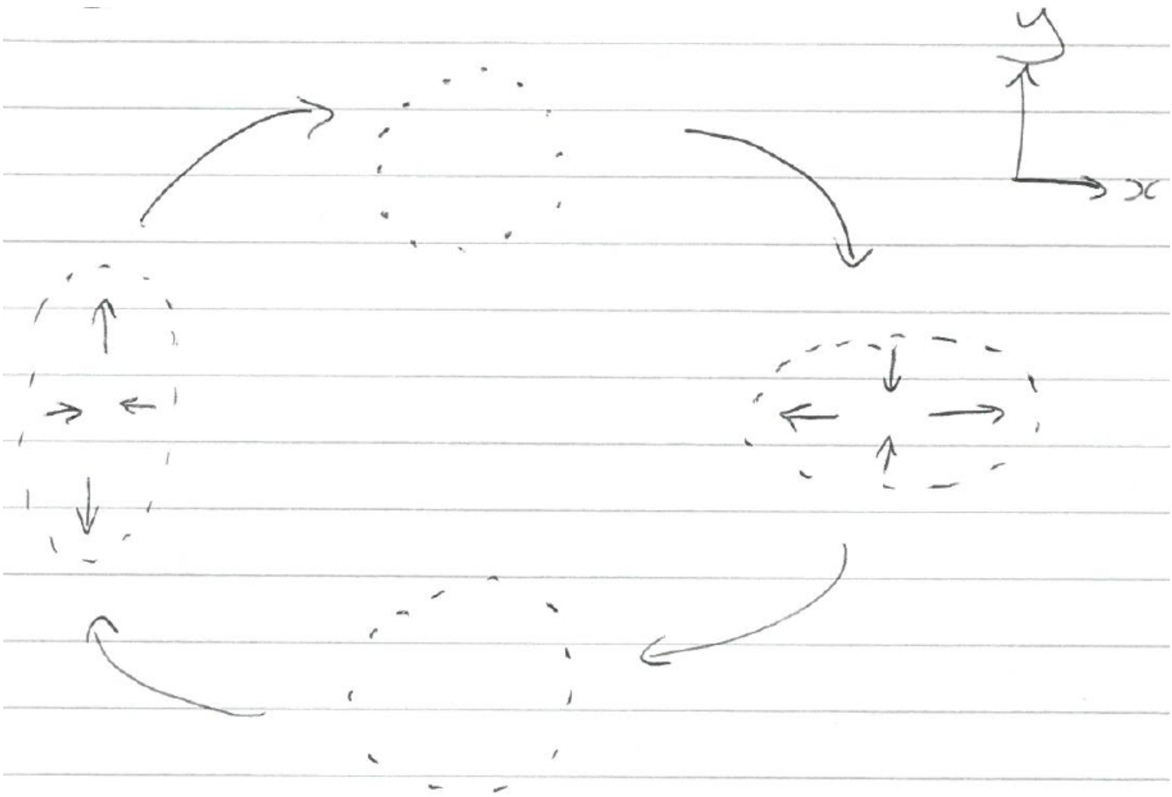
And a proper distance in the z-direction of

$$L_z = \int_0^{\Delta z} \left(1 + \frac{1}{2}h_{zz}\right) dz = \Delta z$$

The effect of the h_+ polarisation is therefore to increase/decrease the separation between test particles in an oscillatory way, in the directions transverse to propagation direction of the wave.

If you imagine a gravitational wave coming upwards, out of the page, the consequences of the h_+ polarisation on a ring of test particles is therefore There is stretching and squashing in the x and y-directions, so that the ring turns into an ellipsoid and back again. The stretching and squashing continues until the wave has passed.

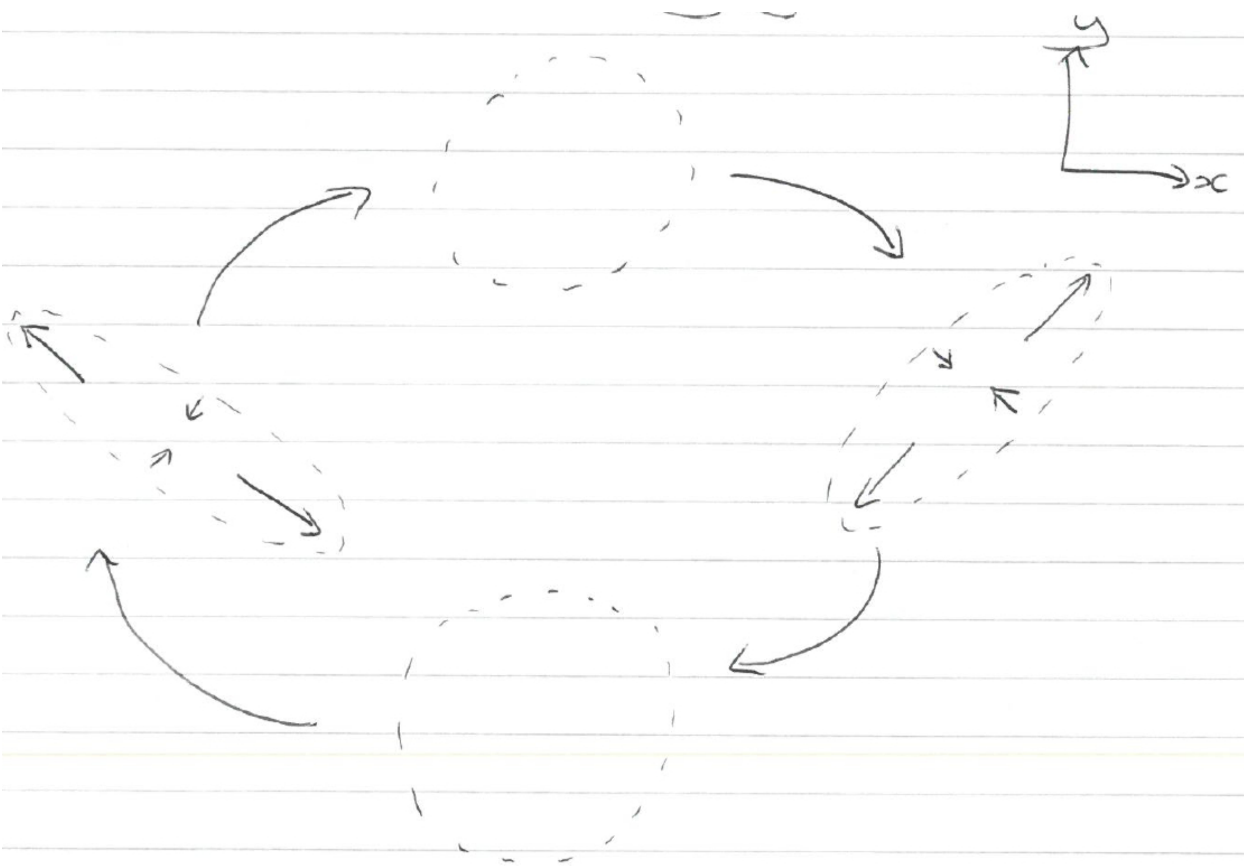
What about the h_x polarisation? A gravitational wave can either be h_+ or h_x polarized, or a mixture of both



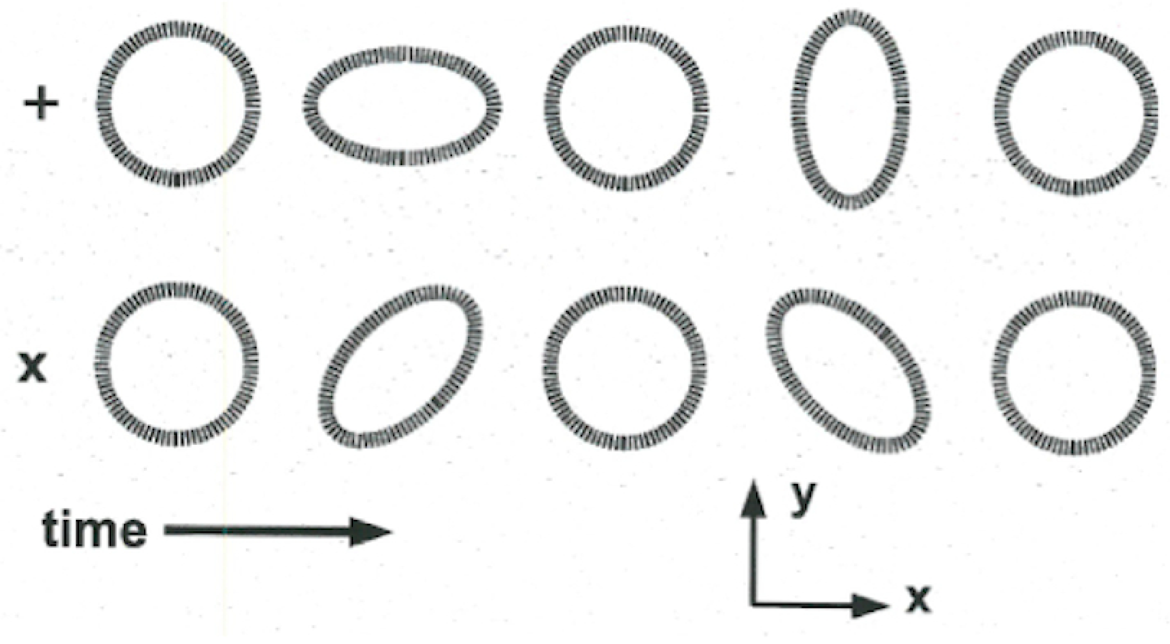
<u>Exercise</u>: By rotating $h^{\mathrm{TT}}_{\mu\nu}$ using the rotation matrix below

$$\frac{\partial x^{\mu}}{\partial x'^{\nu}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and with $\theta = \frac{\pi}{4}$, show that the effect of the $h_{\times}$ polarisation on a ring of particles is given by the image in the figure below.



The effect of the h_+ and $h_\times$ polarisations on a ring of test particles is therefore given in the following figure:



3.7 Linearised solutions with sources

Let us now return to the general equation

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

where $T_{\mu\nu}$ represents an arbitrary distribution of matter. The solution to this equation can be written as

$$\boxed{\bar{h}_{\mu\nu}(t,\bar{x}) = +4G \int \frac{T_{\mu\nu}(t - |\bar{x}-\bar{y}|, \bar{y})}{|\bar{x}-\bar{y}|} d^3y}$$

where $\bar{x}$ is the spatial position where $\bar{h}_{\mu\nu}$ is being evaluated, $\bar{y}$ is a point within the source of the gravitational field, and $|\bar{x}-\bar{y}|$ is the distance between them.

It can be seen that the gravitational perturbation $\bar{h}_{\mu\nu}$ is only sourced by matter that intersects the light cone at the field point $(t, \bar{x})$. This is a very important point: in GR gravitational interactions and disturbances propagate only at the speed of light.

3.8 The quadrupole formula

The quadrupole formula is a very useful result for gravitational wave physics. to derive it we start by making the compact wave source approximation:

$$|\bar{y}| \ll |\bar{x}| \qquad \Rightarrow \qquad |\bar{x} - \bar{y}| \approx |\bar{x}|$$

Our general linearised solution is then

$$\bar{h}_{\mu\nu}(t, \bar{x}) \approx +\frac{4G}{r} \int T_{\mu\nu}(t - r, \bar{y}) d^3y$$

where $r \equiv |\bar{x}|$ is the distance between the source and the observer. If we further assume that the source of the gravitational field is moving much slower than light, $|\bar{v}| \ll c$, then the integral above corresponds to

$$\int T_{00} d^3y \approx \int \bar{T}(\bar{u}, \bar{u}) d^3y \equiv M$$

$$\int T_{0i} d^3y \approx \int \bar{T}(\bar{u}, \bar{x}_i) d^3y \equiv -Q_i$$

$$\int T_{ij} d^3y \approx \int \bar{T}(\bar{x}_i, \bar{x}_j) d^3y \equiv \Pi_{ij}$$

If we now choose our frame of reference to be the centre-of-momentum frame, where $Q_i = 0$, then we are left with

$$\boxed{\bar{h}_{00} = +\frac{4GM}{r}}$$

$$\bar{h}_{ij} = +\frac{4G\Pi_{ij}}{r}$$

$$\bar{h}_{0i} = \bar{h}_{i0} = 0$$

To go further requires manipulating the stress-energy tensor. Taking the leading-order parts of the conservation equations gives

$$\partial_t T^{00} + \partial_i T^{0i} = 0 \tag{3}$$

$$\partial_t T^{i0} + \partial_j T^{ij} = 0 \tag{4}$$

Let us now consider the integral

$$\int \partial_k (T^{ik} y^j) d^3y = \int (\partial_k T^{ik}) y^j d^3y + \int T^{ij} d^3y$$

Using Gauss' divergence result, the integral on the LHS can be transformed to a surface integral over the boundary of the original domain of integration. If the domain of integration is larger than our compact source (which it is) then it must vanish, as $T_{\mu\nu}$ is only non-zero inside the source.

This means

$$\int T^{ij} d^3y = -\int (\partial_k T^{ik} y^j d^3y)$$

then using equation (2)

$$\int T^{ij} d^3y = \int (\partial_t T^{i0}) y^j d^3y = \frac{d}{dt} \int T^{i0} y^j d^3y$$

Similarly, exchanging i and j indices, gives

$$\int T^{ji} d^3y = \frac{d}{dt} \int T^{j0} y^i d^3y$$

so

$$\int T^{ij} d^3y = \frac{1}{2} \frac{d}{dt} \int (T^{i0} y^j + T^{j0} y^i) d^3y \tag{5}$$

Now, consider a new integral

$$\int \partial_k (T^{0k} y^i y^j) d^3y = \int (\partial_k T^{0k}) y^i y^j d^3y + \int (T^{0i} y^j + T^{0j} y^i) d^3y$$

$$= -\int (\partial_t T^{00}) y^i y^j d^3y + \int (T^{0i} y^j + T^{0j} y^i) d^3y$$

$$= -\frac{d}{dt} \int T^{00} y^i y^j d^3y + \int (T^{0i} y^j + T^{0j} y^i) d^3y$$

Once again, the LHS of this equation can be set to zero using Gauss' result. This gives

$$\int (T^{0i}y^j + T^{0j}y^i)d^3y = \frac{d}{dt}\int T^{00}y^iy^jd^3y \tag{6}$$

Now, substituting equation (4) into equation (3) gives

$$\int T^{ij}d^3y = \frac{1}{2}\frac{d^2}{dt^2}\int T^{00}y^iy^jd^3y$$

The LHS of this equation is identical to our integral pressure term Π_{ij} (after lowering indices with η_{ij}). We can therefore write $\bar{h}_{ij}$ as

$$\boxed{\bar{h}_{ij} = +\frac{2G}{r}\frac{d^2I_{ij}}{dt^2}}$$

where $I_{ij} = \int \rho y^iy^jd^3y$ is the quadrupole moment tensor. The above equation is the "quadrupole formula". It forms the basis of much of gravitational wave physics.

3.9 Static sources and Newtonian limit

In the centre of momentum frame, if the source of the gravitational field is static (not moving) then

$$\frac{d^2 I_{ij}}{dt^2} = 0$$

The only non-vanishing part of $\bar{h}_{\mu\nu}$ is then

$$\bar{h}_{00} = +\frac{4GM}{r}$$

This means $\bar{h}^{\mu}{}_{\mu} = -\frac{4GM}{r}$, and therefore

$$h_{00} = \bar{h}_{00} - \frac{1}{2}\eta_{00}\bar{h}^{\mu}{}_{\mu} = +\frac{2GM}{r}$$

$$h_{ij} = \bar{h}_{ij} - \frac{1}{2}\eta_{ij}\bar{h}^{\mu}{}_{\mu} = +\frac{2GM}{r}\delta_{ij}$$

$$\Rightarrow ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 + \frac{2GM}{r}\right)(dx^2 + dy^2 + dz^2)$$

This line-element captures enough to describe Newtonian gravity, and the leading-order part of the bending of light. It can also be derived by assuming that the gravitational field's source is dominated by its rest mass, i.e.

$$|T_{00}| \gg |T_{0i}| \qquad \text{and} \qquad |T_{00}| \gg |T_{ij}|$$

3.10 Waves from binary systems

Consider two massive bodies, in a circular orbit around each other of radius R:

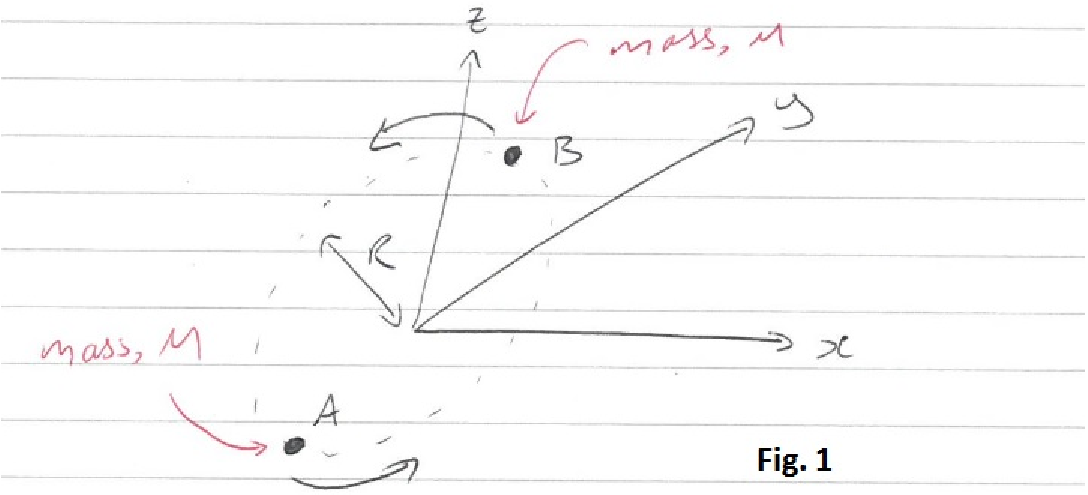


Fig. 1

By rotating spatial coordinates we can arrange for these bodies to orbit in the plane $x = 0$ (as above). The leading order part of the gravitational field is Newtonian, so this orbit obeys

$$\frac{Mv^2}{R} = \frac{GM^2}{(2R)^2}$$

If we now write $v = \Omega R$, then we get angular speed

$$\Omega = \sqrt{\frac{GM}{4R^3}}$$

The positions of the two bodies can also be written

$$x^i_A = \big(0, R\cos(\Omega t), R\sin(\Omega t)\big)$$

and

$$x^i_B = \big(0, -R\cos(\Omega t), -R\sin(\Omega t)\big)$$

The density of this system can therefore be written as

$$\rho - M\delta(x)[\delta(y - R\cos\Omega t)\delta(z - R\sin\Omega t) + \delta(y + R\cos\Omega t)\delta(z + R\sin\Omega t)]$$

<u>Exercise</u>: substitute this expression for ρ into $I^{ij} = \int \rho y^i y^j d^3y$ to find

$$I^{ij} = MR^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 + \cos 2\Omega t & \sin 2\Omega t \\ 0 & \sin 2\Omega t & 1 - \cos 2\Omega t \end{pmatrix}$$

Solution: we immediately have

$$I^{xx} = \int \rho x^2 dxdydz = 0$$

$$I^{xy} = I^{yx} = \int \rho xydxdydz = 0$$

$$I^{xz} = I^{zx} = \int \rho xzdxdydz = 0$$

$$I^{yy} = \int \rho y^2 dxdydz = M \int \big[y^2\delta(y - R\cos\Omega t) + y^2\delta(y + R\cos\Omega t)\big]dy$$

$$= M(R^2\cos^2\Omega t + R^2\cos^2\Omega t) = MR^2(1 + \cos 2\Omega t)$$

$$I^{zz} = \int \rho z^2 dxdydz = M \int \big[z^2\delta(z - R\sin\Omega t) + z^2\delta(z + R\sin\Omega t)\big]dz$$

$$= M(R^2\sin^2\Omega t + R^2\sin^2\Omega t) = MR^2(1 - \cos 2\Omega t)$$

$$I^{yz} = I^{zy} = \int \rho yzdxdydz$$

$$= M \int yz\delta(y - R\cos\Omega t)\delta(z - R\sin\Omega t)dydz$$

$$+ M \int yz\delta(y + R\cos\Omega t)\delta(z + R\sin\Omega t)dydz$$

$$= M(R\cos\Omega t \cdot R\sin\Omega t) + M\big((-R\cos\Omega t)\cdot(-R\sin\Omega t)\big)$$

$$= 2MR^2\cos\Omega t\sin\Omega t = MR^2\sin 2\Omega t$$

This all can be written as follows:

$$\Rightarrow I^{ij} = MR^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1+\cos 2\Omega t & \sin 2\Omega t \\ 0 & \sin 2\Omega t & 1-\cos 2\Omega t \end{pmatrix}$$

Substituting into the quadrupole formula gives

$$\bar{h}_{ij} = -\frac{8GMR^2\Omega^2}{r} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos 2\Omega(t-r) & \sin 2\Omega(t-r) \\ 0 & \sin 2\Omega(t-r) & -\cos 2\Omega(t-r) \end{pmatrix}$$

3.11 Gravity of gravitational waves

Gravitational waves do not have any local energy or momentum (i.e. they do not contribute to $T_{\mu\nu}$). However, they do have their own gravitational field. This requires some careful thought to be properly understood.

Recall that in section 3.1 we wrote down (but did not yet use) $R_{\mu\nu}$ to order h^2. If we do the same with the Einstein tensor then we can write

$$G_{\mu\nu} = G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + \ldots = 8\pi G T_{\mu\nu}$$

If we move $G^{(2)}_{\mu\nu}$ to the RHS we have

$$G^{(1)}_{\mu\nu} = 8\pi G(T_{\mu\nu} + t_{\mu\nu}) + O(h^3)$$

where $t_{\mu\nu} \equiv -\frac{1}{8\pi G}G^{(2)}_{\mu\nu}$. When written in this form we see that $t_{\mu\nu}$ acts as a source term for the leading order part of the gravitational field (albeit a small one). The second-order part of the Einstein tensor is given explicitly by

$$G^{(2)}_{\mu\nu} = R^{(2)}_{\mu\nu} - \frac{1}{2}(\eta_{\mu\nu}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma} + h_{\mu\nu}\eta^{\rho\sigma}R^{(1)}_{\rho\sigma} - \eta_{\mu\nu}h^{\rho\sigma}R^{(1)}_{\rho\sigma})$$

where $R^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ are given in section 3.1. This quantity is not by itself a tensor, as can be verified by trying to perform a coordinate transformation. However, it can be made into a tensor by integrating (or smoothing) it over a small region of spacetime. This gives

$$\begin{aligned}\langle t_{\mu\nu}\rangle = \frac{1}{32\pi G}\Big\langle(\partial_\mu \bar{h}_{\rho\sigma})\partial_\nu \bar{h}^{\rho\sigma} - (\partial_\sigma \bar{h}^{\rho\sigma})\partial_\mu \bar{h}_{\nu\rho} \\ -(\partial_\sigma \bar{h}^{\rho\sigma})\partial_\nu \bar{h}_{\mu\rho} - \frac{1}{2}(\partial_\mu \bar{h})(\partial_\nu \bar{h})\Big\rangle \\ -\frac{1}{4}\big\langle 2\bar{h}_{\rho\nu}T^{\rho}{}_{\nu} + 2\bar{h}_{\rho\nu}T^{\rho}{}_{\nu} + \eta_{\mu\nu}h^{\rho\sigma}T_{\rho\sigma}\big\rangle\end{aligned}$$

where $\langle \ldots \rangle$ denotes the smoother quantity, and where use has been made of the result $\langle \partial_\nu \Im \rangle = 0$ for any function $\Im = \Im(x^\mu)$.

3.12 Energy radiated from a binary

We can now use $\langle t_{\mu\nu}\rangle$ to work out the rate at which binary systems lose energy through the emission of gravitational waves. In vacuum, and in TT gauge, we get

$$\langle t_{\mu\nu}\rangle = \frac{1}{32\pi G}\langle(\partial_\mu h^{TT}_{\rho\sigma})(\partial_\nu h^{TT\rho\sigma})\rangle$$

The energy flux in gravitational waves is therefore

$$q_i^{\mathrm{GW}} = -\langle t_{0i}\rangle$$

and the rate of energy loss from the system emitting them is

$$\frac{dE}{dt} = -\oint r^2 q_i^{GW}\hat{r}^i d\Omega$$

where the integration is over a sphere that contains the system at the centre, and where $\hat{r}^i$ are the spatial components of an outward pointing radial unit vector (such that $\hat{r}^i\hat{r}_i = 1$).

We will now use the TT part of the quadrupole formula

$$h^{ij}_{\mathrm{TT}} = \bar{h}^{ij}_{\mathrm{TT}} = (P^i{}_j P^j{}_l - \frac{1}{2}P^{ij}P_{kl})\frac{2G}{r}\frac{d^2 I^{kl}}{dt^2} = \frac{2G}{r}\frac{d^2 I^{ij}_{\mathrm{TT}}}{dt^2}$$

where $I^{ij}_{\mathrm{TT}} \equiv (P^i{}_k P^j{}_l - \frac{1}{2}P^{ij}P_{kl})I^{kl}$. This gives

$$\partial_t h^{ij}_{\mathrm{TT}} = \frac{2G}{r} \dddot{I}^{ij}_{\mathrm{TT}}$$

and

$$\partial_r h^{ij}_{\mathrm{TT}} = -\frac{2G}{r^2} \ddot{I}^{ij}_{\mathrm{TT}} - \frac{2G}{r} \dddot{I}^{ij}_{T} \approx -\frac{2G}{r} \dddot{I}^{ij}_{\mathrm{TT}}$$

Substituting this all back into the equations above gives

$$\frac{dE}{dt} = -\oint \frac{G}{8\pi} \langle \dddot{I}^{TT}_{ij} \dddot{I}^{ij}_{TT} \rangle d\Omega$$

Finally, the term in brackets can be expanded as

$$\dddot{I}^{\mathrm{TT}}_{ij} \dddot{I}^{ij}_{\mathrm{TT}} = \dddot{I}_{ij} \dddot{I}^{ij} - 2 \dddot{I}_i{}^j \dddot{I}^{ik} \hat{r}_j \hat{r}_k + \frac{1}{2} \dddot{I}^{ij} \dddot{I}^{kl} \hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_l$$

Using the known results

$$\oint d\Omega = 4\pi, \qquad \oint \hat{r}_i \hat{r}_j d\Omega = \frac{4\pi}{3} \delta_{ij}$$

$$\oint \hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_i d\Omega = \frac{4\pi}{15} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

then gives

$$\boxed{\frac{dE}{dt} = -\frac{G}{5} \langle \dddot{I}_{ij} \dddot{I}^{ij} \rangle}$$

Example: consider the binary system from section 3.10, for which

$$\dddot{I}^{ij} = 8MR^2\Omega^3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin 2\Omega t & -\cos 2\Omega t \\ 0 & -\cos 2\Omega t & -\sin 2\Omega t \end{pmatrix}$$

This gives

$$\Rightarrow \dddot{I}_{ij}\dddot{I}^{ij} = 128M^2R^4\Omega^6$$

$$\Rightarrow \boxed{\frac{dE}{dt} = -\frac{128}{5}GM^2R^4\Omega^6}$$

This result is very important for binary pulsar observations, and the recent gravitational wave detection by LIGO.

3.13 The Hulse-Taylor binary

In 1993 Russell Hulse and Joseph Taylor were awarded the Nobel Prize in Physics, for their work on the binary system PSR B1913+16. This was a system of two neutron stars, one of which was a pulsar, that allowed evidence for the existence of gravitational waves to be inferred. To consider why and how, let's return to our two bodies of mass M in circular orbit.

The total energy of the system is

$$E = K + U = 2 \times \left(\frac{1}{2}Mv^2\right) - \frac{GM^2}{2R} = -\frac{GM^2}{4R}$$

as we know $v^2 = \frac{GM}{4R}$ from before. Using $v = \Omega R$ and differentiating

$$\Rightarrow \frac{dE}{dt} = -\frac{2\dot{\Omega}}{3}\Big(\frac{G^2M^5}{16\Omega}\Big)^{1/3}$$

From the previous section we now know that the emission of gravitational waves from such a system causes

$$\frac{dE}{dt} = -\frac{128}{5}GM^2\Omega^6\Big(\frac{GM}{4\Omega^2}\Big)^{4/3}$$

Equating these two expressions gives the change in angular velocity due to GW emission:

$$\dot{\Omega} = \frac{48}{5} \times 2^{2/3} \times (GM)^{5/3}\Omega^{11/3}$$

which can be integrated to give

$$\Rightarrow \Omega^{-8/3} = \frac{128}{5} \times 2^{2/3} \times (GM)^{5/3}(t_0 - t)$$

where t_0 is an integration constant (the time when $\Omega \to \infty$, when the two bodies eventually coalesce).

Now the period of the orbit is

$$\boxed{\tau = \frac{2\pi}{\Omega} = \frac{2^{31/8}}{5^{3/8}}\pi(GM)^{5/8}(t-t_0)^{3/8}}$$

This result gives the rate at which our systems period decreases due to energy lost through gravitational radiation, $\tau \propto (t_0 - t)^{3/8}$. This is a very good match to the observations made by Hulse and Taylor for PSR B1913+16, and is widely considered to be the first indirect evidence for the existence for gravitational waves.

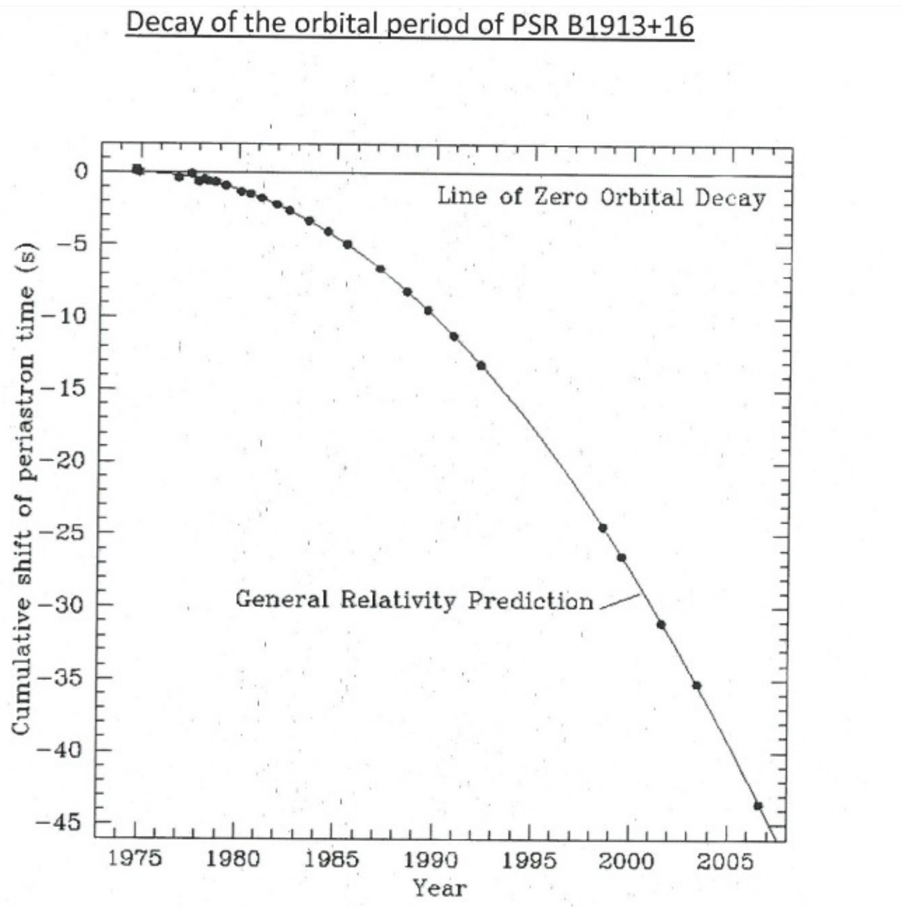


Fig. 2

3.14 The LIGO detections

The first direct detection of GWs was made by the LIGO experiment, on the 14th of September 2015. This experiment consists of two interferometers at two different locations in the USA

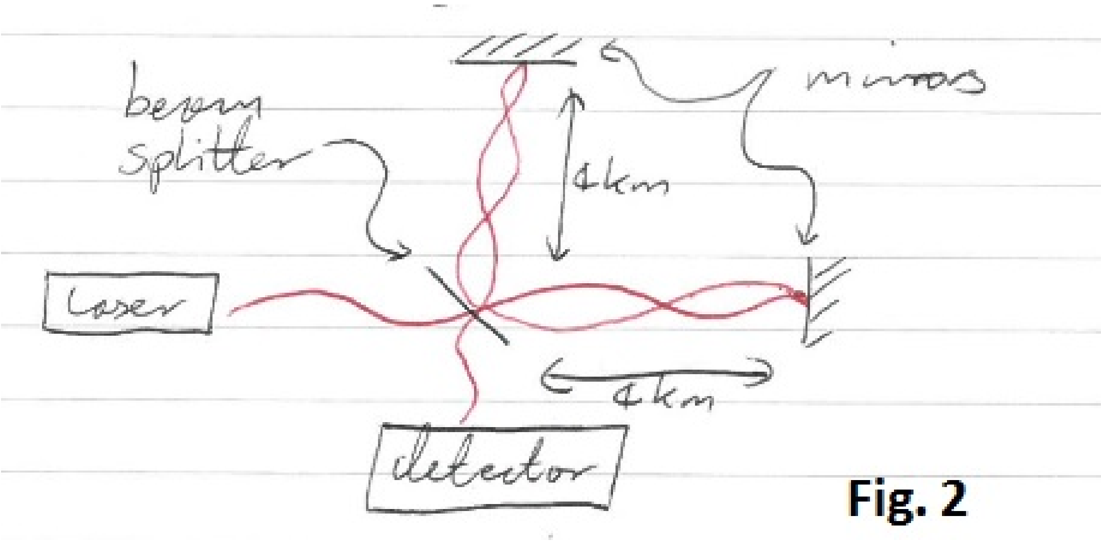


Fig. 2

The idea is this: when a gravitational wave passes, the two arms of the detectors change length by a small amount. This causes a change in the interference pattern at the detector. This sounds simple, but gravitational waves tend to have very low amplitude (the 2015 detection caused the arms to change length by about 10^{-18}m). To make a positive detection therefore a very careful experimentation, and some knowledge of the signal that is expected.

Let's return to our two bodies in a circular orbit. At the end of section 3.10 we found an explicit expression for $\bar{h}_{ij}$, for the emitted gravitational waves. Now, because we have an explicit expression for $\Omega = \Omega(t)$ in section 3.13, we can work out what an observer at some position r on the z-axis should be expected to see with his/her gravitational wave detector.

Example: use the results from Section 3.5 of the notes to show that waves travelling in the z-direction from this system can be writte in TT gauge as

$$\bar{h}^{\mathrm{TT}}_{\mu\nu} = \frac{(GM)^{\frac{5}{3}}}{r}(2\Omega)^{\frac{2}{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos 2\Omega(t-r) & 0 & 0 \\ 0 & 0 & -\cos 2\Omega(t-r) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence correspond to a wave with + polarization and

$$h_+ = \frac{(GM)^{\frac{5}{3}}}{r}(2\Omega)^{\frac{2}{3}} \cos 2\Omega(t-r).$$

Solution: for the system in question we have

$$\bar{h}_{ij} = -\frac{8GMR^2\Omega^2}{r}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos 2\Omega(t-r) & \sin 2\Omega(t-r) \\ 0 & \sin 2\Omega(t-r) & -\cos 2\Omega(t-r) \end{pmatrix}$$

To put this in TT gauge recall $\bar{h}^{\mathrm{TT}}_{ij} = (P_i{}^k P_j{}^l - \frac{1}{2}P_{ij}P^{kl})\bar{h}_{kl}$, where $P_i{}^j = \delta_i{}^j - \hat{z}_i\hat{z}^j$ and $\hat{z}^i = (0,0,1)$.

$$\Rightarrow \qquad P_i{}^k P_j{}^l \bar{h}_{kl} = -\frac{8GMR^2\Omega^2}{r}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos 2\Omega(t-r) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\qquad P^{kl}\bar{h}_{kl} = -\frac{8GMR^2\Omega^2}{r}\cos 2\Omega(t-r)$

$$\Rightarrow \qquad -\frac{1}{2}P_{ij}P^{kl}\bar{h}_{kl} = \frac{8GMR^2\Omega^2}{r}\begin{pmatrix} \frac{1}{2}\cos 2\Omega(t-r) & 0 & 0 \\ 0 & \frac{1}{2}\cos 2\Omega(t-r) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

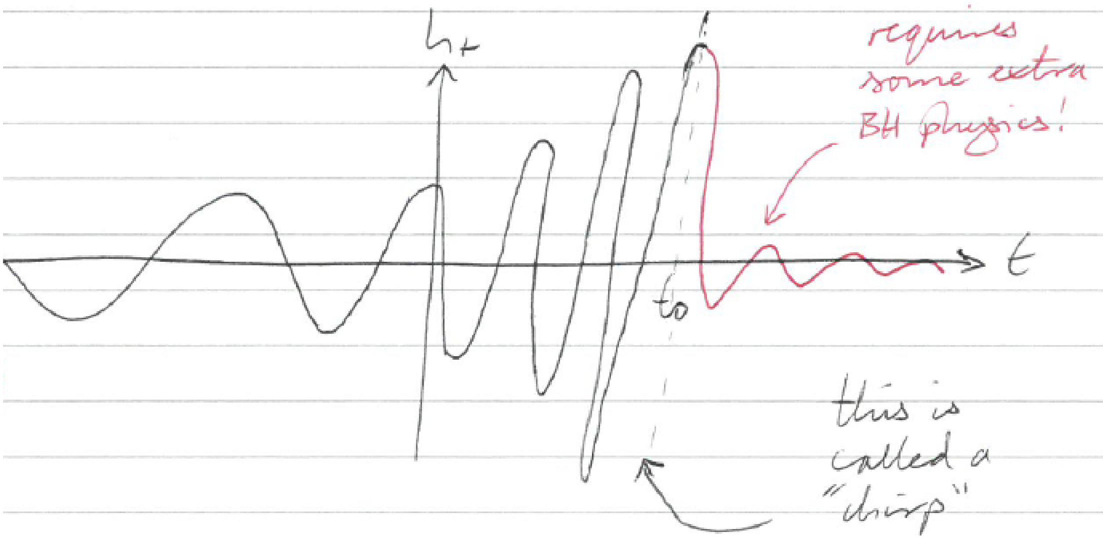
$$\Rightarrow \qquad \bar{h}^{\mathrm{TT}}_{ij} = \frac{4GMR^2\Omega^2}{r}\begin{pmatrix} \cos 2\Omega(t-r) & 0 & 0 \\ 0 & -\cos 2\Omega(t-r) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This gives the desired result when we use $\Omega^2 = \frac{GM}{4R^2}$, or $R^2 = \left(\frac{GM}{4\Omega^2}\right)^{\frac{2}{3}}$.

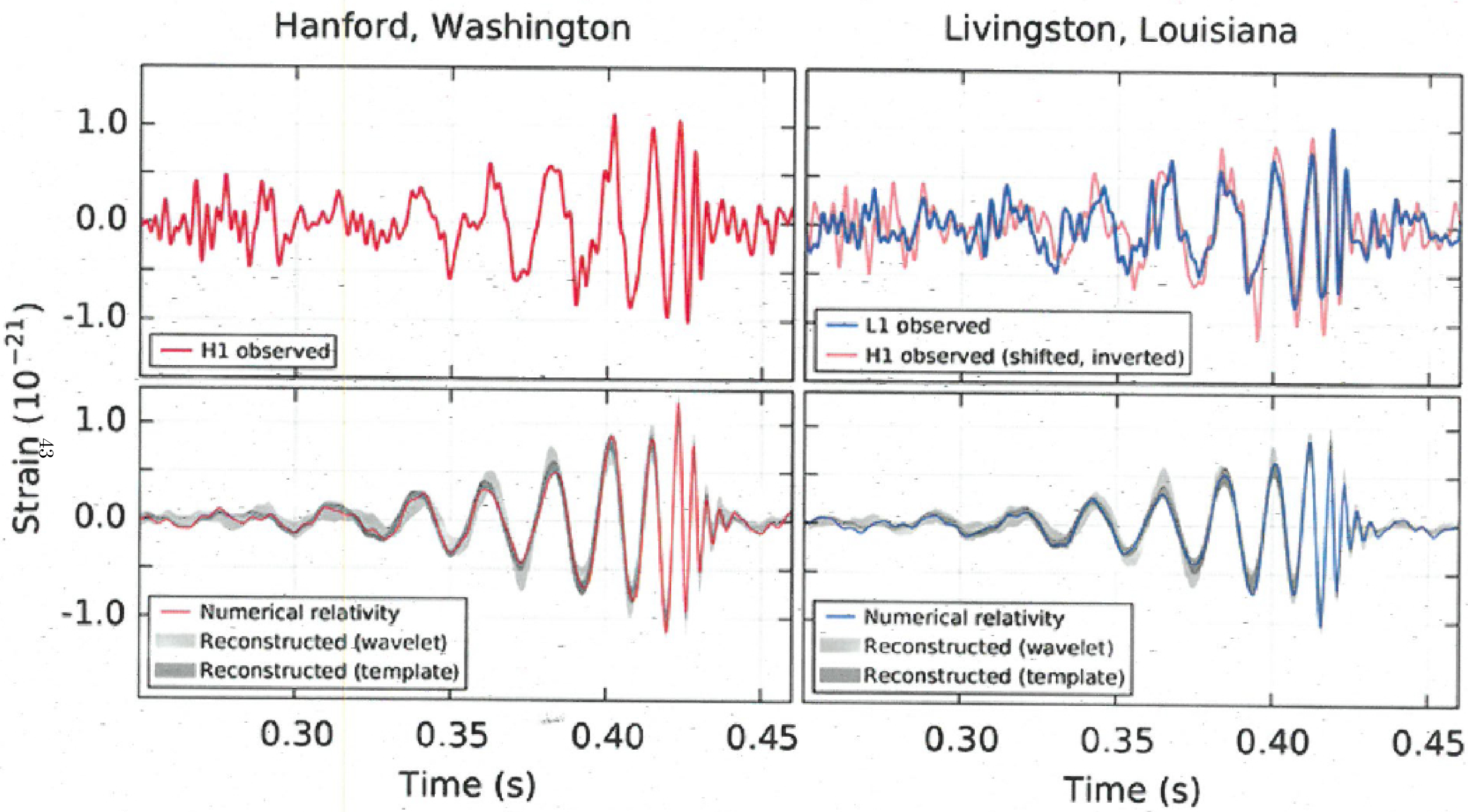
Now, because we have an explicit expression for $\Omega = \Omega(t)$ in Section 3.13, we can work out what an observer at some position r on the z-axis should be expected to see with his/her gravitational wave detector.

Exercise: produce some plots of $h_+ = h_+(t)$ to see what a gravitational wave signal looks like.

You should get something that looks a bit like this:



The frequency and amplitude of the wave increases as the bodies come together, at $t = t_0$. This is exactly what was seen by LIGO: two merging black holes, each with about thirty times the mass of the Sun, as at a distance of about 1.4 billion light years. The actual signal from the first LIGO detection is shown on the next page, for the two detectors at Hanford and Livingston.


Hanford, Washington
Livingston, Louisiana
Strain (10^{-21})
1.0
0.0
-1.0
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
0.30
0.35
0.40
0.45
Time (s)