RELATIVITY & GRAVITATION

Lectured by Timothy Clifton

SECTION 2 - EINSTEIN'S THEORY AND BLACK HOLES

2.1 Observables

Coordinates are not observable. The components of vectors and tensors in a coordinate basis are not (by themselves) observable, because they depend on the choice of coordinates.

Things that are observable:

- proper time between two events on a timelike curve, i.e. a clock
- proper distances between spacelike separated events, i.e. a stick length
- scalar quantities at a point p, i.e. the Ricci scalar, R
- the frame components of vectors and tensors

The first three of these should be familiar. The fourth is new, and is of fundamental importance in GR. Most things fall into the fourth category.

2.2 Observers and frames

In a general spacetime an observer will follow some timelike world line $x^\mu(\tau)$ (we use proper time τ as the parameter). The tangent vector to this world line is

$$\vec{u} = \frac{dx^\mu}{d\tau}\vec{e}_\mu$$

This is the 4-velocity of the observer, and obeys $\vec{u} \cdot \vec{u} = -1$ as we have chosen the parameter to measure proper time along the curve. In general time-like curves have tangent vectors that obey $\vec{u} \cdot \vec{u} < 0$.

We can use $\vec{u}$ as one of the basis vectors at each point along C. The rest-space of the observer that follows C will then be spanned by three vectors that are orthogonal to $\vec{u}$: i.e. $\vec{x}_i$ such that $\vec{x}_i \cdot \vec{u} = 0$, where index $i = 1, 2, 3$. If we chose the parameter along each $\vec{x}_i$ to be proper distance then we have

$$\vec{x}_i \cdot \vec{x}_i = 1$$

In general space-like curves have tangent vectors that obey $\vec{x}_i \cdot \vec{x}_i > 0$. *Note: no sum over i is implied in these equations!*

If we choose the three spacelike vectors to be mutually orthogonal then this becomes

$$\vec{x}_i \cdot \vec{x}_j = \delta_{ij} \, .$$

We now have three linearly independent unit vectors in the rest-space of the observer with 4-velocity $\vec{u}$ following C. These are like the $\{\hat{x}, \hat{y}, \hat{z}\}$

vectors from Euclidean geometry, but in this case are unique to our specific observer. If we collect $\vec{u}$ and the $\vec{x}_i$ together then we can collectively label them $\hat{e}_a$, such that

$$\hat{\vec{e}}_0 = \vec{u}, \qquad \hat{\vec{e}}_1 = \vec{x}_1, \qquad \hat{\vec{e}}_2 = \vec{x}_2 \qquad \hat{\vec{e}}_3 = \hat{x}_3$$

This set of 4 vectors obey the relation

$$\boxed{\hat{\vec{e}}_a \cdot \hat{\vec{e}}_b = \eta_{ab}},$$

where η_{ab} is the Minkowski metric, familiar from SR. Note that the above are not written using coordinate indices because we are not working with coordinates, $\hat{\vec{e}}_a$ vectors are "frame vectors" since they define a frame. In GR, the 4 vectors that made up the basis $\hat{\vec{e}}_a$ are referred to as a frame, for the observer following the world line C

Example: write down a set of frame vectors for an observer at constant r, θ and ϕ in this geometry:

$$ds^2 = -\left(1 - \frac{2Gm}{r}\right) dt^2 \frac{dr^2}{\left(1 - \frac{2Gm}{r}\right)} + r^2 \left(d\theta^2 \sin^2\theta d\phi^2\right) \; .$$

Solution: Let's write the frame vectors as $\vec{u}$, $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$. If the observer stays at fixed (r, θ, ϕ) then only the t-component of $\vec{u}$ can be non-zero, so

$$u^\mu u_\mu = (u^t)^2 g_{tt} = -\left(1 - \frac{2Gm}{r}\right)(u^t)^2 = -1$$

which means

$$\Rightarrow \quad u^t = \frac{1}{\sqrt{1 - \frac{2Gm}{r}}} \qquad \Rightarrow \quad \vec{u} = \frac{1}{\sqrt{1 - \frac{2Gm}{r}}}\,\vec{e}_t\,.$$

If we now take $\vec{x}_1$ to point in the r-direction then

$$x_1^\mu x_{1\mu} = (x_1^r)^2 g_{rr} = \frac{(x_1^r)^2}{1 - \frac{2Gm}{R}} = 1$$

which means

$$\Rightarrow \quad x_1^r = \sqrt{1 - \frac{2Gm}{r}} \qquad \Rightarrow \quad \vec{x}_1 = \sqrt{1 - \frac{2Gm}{r}}\,\vec{e}_r\,.$$

The final two vectors can then be written as

$$\vec{x}_2 = \frac{1}{r}\,\vec{e}_\theta \qquad \text{and} \qquad \vec{x}_3 = \frac{1}{r\sin\theta}\,\vec{e}_\phi$$

so $\vec{x}_2 \cdot \vec{x}_2 = 1 = \vec{x}_3 \cdot \vec{x}_3$ and $\vec{x}_1 \cdot \vec{x}_2 = \vec{x}_1 \cdot \vec{x}_3 = \vec{x}_2 \cdot \vec{x}_3 = \vec{u} \cdot \vec{x}_1 = \vec{u} \cdot \vec{x}_2 = \vec{u} \cdot \vec{x}_3 = 0$, as required.

2.3 Frame transformations

The frame we just introduced is required to satisfy $\hat{\vec{e}}_a \cdot \hat{\vec{e}}_b = \eta_{ab}$. This equation by itself, however, is not sufficient to fix $\hat{\vec{e}}_a$ uniquely. There are freedoms that remain. If we keep the observer fixed, so that $\vec{u}$ does not change, then there is a group of transformations that we can perform on $\vec{x}_i$ that leave $\vec{x}_i \cdot \vec{x}_j = \delta_{ij}$ unchanged. Consider

$$\vec{x}_i = M^{j'}{}_i \vec{x}_{j'}, \qquad i, j = 1, 2, 3$$

$$\Rightarrow \vec{x}_i \cdot \vec{x}_j = M^{k'}{}_i M^{l'}{}_k \vec{x}_{k'} \cdot \vec{x}_{l'} = M^{k'}{}_i M^{l'}{}_j \delta_{k'l'}$$

so if $M^{k'}{}_i M^{l'}{}_j \delta_{k'l'} = \delta_{ij}$ then both $\vec{x}_i$ and $\vec{x}_{i'}$, obey the required condition. This makes $M^{j'}{}_i$ an "orthogonal" transformation, which can encode either reflections or rotations of the spatial frame vectors

Example: rotation about $\vec{x}_3$

$$\{M^{j'}{}_i\} = \begin{pmatrix} M^{1'}{}_1 & M^{1'}{}_2 & M^{1'}{}_3 \\ M^{2'}{}_1 & M^{2'}{}_2 & M^{2'}{}_3 \\ M^{3'}{}_1 & M^{3'}{}_2 & M^{3'}{}_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \qquad \begin{aligned} \vec{x}_1 &= \cos\theta x_{1'} - \sin\theta \vec{x}_{2'} \\ \vec{x}_2 &= \sin\theta \vec{x}_{1'} + \cos\theta \vec{x}_{2'} \\ \vec{x}_3 &= \vec{x}_{3'} \end{aligned}$$

Now consider what happens if we change frames between two different observers who are in relative motion and both at the same point in spacetime. Their frame vectors exist in the same tangent space, and must obey

$$\hat{\vec{e}} \cdot \hat{\vec{e}}_b = \eta_{ab} \qquad \text{and} \qquad \hat{\vec{e}}'_a \cdot \hat{\vec{e}}'_b = \eta_{ab}$$

These two sets of vectors must be linearly related, so we can write

$$\hat{\vec{e}}_a = L^{b'}{}_a \hat{\vec{e}}'_b, \qquad \text{for some} \quad L^{b'}{}_a$$

$$\Rightarrow \hat{\vec{e}}_a \cdot \hat{\vec{e}}_b = L^{c'}{}_a L^{d'}{}_b \hat{\vec{e}}'_c \cdot \hat{\vec{e}}'_d = L^{c'}{}_a L^{d'}{}_b \eta_{c'd'}$$

$$\Rightarrow L^{c'}{}_a L^{d'}{}_b \eta_{c'd'} = \eta_{ab}$$

The $L^{b'}{}_a$ that obeys this condition are the Lorentz transformations (which encode boosts and rotations).

Example: boost in the x-direction

$$\{L^{b'}{}_a\} = \begin{pmatrix} L^{0'}{}_0 & L^{0'}{}_1 & L^{0'}{}_2 & L^{0'}{}_3 \\ L^{1'}{}_0 & L^{1'}{}_1 & L^{1'}{}_2 & L^{1'}{}_3 \\ L^{2'}{}_0 & L^{2'}{}_1 & L^{2'}{}_2 & L^{2'}{}_3 \\ L^{3'}{}_0 & L^{3'}{}_1 & L^{3'}{}_2 & L^{3'}{}_3 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$$\begin{aligned} \Rightarrow \qquad \vec{u} &= 7\gamma\vec{u}' + \gamma\beta\vec{x}'_1 = \gamma(\vec{u}' + \beta\vec{x}'_1) \\ \vec{x}_1 &= \gamma\beta\vec{u}' + \gamma\vec{x}'_1 = \gamma(\vec{x}'_1 + \beta\vec{u}') \\ \vec{x}_2 &= \vec{x}'_2 \\ \vec{x}_3 &= \vec{x}'_3 \end{aligned}$$

These are the familiar expressions for relating time and space directions of two observes in relative motion. They show how time dilation and length contraction from SR should be understood in GR.

Finally, note that (as with any set of basis vectors) we can define a dual according to

$$\hat{\vec{e}}^a \cdot \hat{\vec{e}}_b = \delta^a{}_b$$

It is the frame components of vectors and tensors that are direct observables:

Examples:

$$\begin{aligned} \vec{v} &= v^a \hat{\vec{e}}_a \\ \vec{t} &= t^{ab} \hat{\vec{e}}_a \otimes \hat{\vec{e}}_b \end{aligned}$$

These v^a and t^{ab} are coordinate independent, but do depend on the state of motion of the observer for whom this frame is defined (i.e. they depend on $\vec{u}$). They are observables.

2.4 Lorentz transformations

Frame vectors transform under a Lorentz transformation as $\hat{\vec{e}}_a = L^{b'}{}_a \hat{\vec{e}}_{b'}$, but how do frame components of vectors and tensors transform?

$$\vec{v} = v^a \hat{\vec{e}}_a = v^a L^{b'}{}_a \hat{\vec{e}}_{b'}$$

and

$$\vec{v} = v^{b'} \hat{\vec{e}}_{b'}$$

$$\Rightarrow v^{b'} = L^{b'}{}_{a} v^{a}$$

Similarly,

$$\vec{t} = t^{ab} \hat{\vec{e}}_a \otimes \hat{\vec{e}}_b = t^{ab} (L^{c'}{}_{a} \hat{\vec{e}}_{c'}) \otimes (L^{d'}{}_{b} \hat{\vec{e}}_{d'}) = t^{ab} L^{c'}{}_{a} L^{d'}{}_{b} \hat{\vec{e}}' \otimes \hat{\vec{e}}'_d$$

and

$$\vec{t} = t^{c'd'} \hat{\vec{e}}'_c \otimes \hat{\vec{e}}'_d$$

$$\Rightarrow t^{a'b'} = L^{a'}{}_{c} L^{b'}{}_{d} t^{cd}$$

General rule: each raised index on a frame component requires on a frame component requires one $L^{a'}{}_{b}$ to Lorentz transform

2.5 Light rays

The invariance of the speed of light is fundamental to both SR and GR. In GR, this means that any given frame we must measure $c = 1$. For a photon moving in the $+\vec{x}_1$ direction, this means we should be able to write the tangent vector to its path through spacetime as:

$$\vec{k} = \alpha(\vec{u} + \vec{x}_1)$$

where α is a constant, i.e. the photon increases its location in time along $\vec{u}$ at the same rate it increases its position in space (along $\vec{x}_1$).

$$\Rightarrow \vec{k} \cdot \vec{k} = \alpha^2(\vec{u} + \vec{x}_1) \cdot (\vec{u} + \vec{x}_1)$$

$$= \alpha^2(\vec{u} \cdot \vec{u} + \vec{x}_1 \cdot \vec{u} + \vec{u} \cdot \vec{x}_1 + \vec{x}_1 \cdot \vec{x}_1)$$

$$= \alpha^2(-1 + 0 + 0 + 1)$$

$$= 0$$

This property means that $\vec{k}$ is called a "null vector". In general, null vectors have $\vec{k} \cdot \vec{k} = 0$ (c.f. timelike and spacelike vectors).

Excercise: Prove that the distance along any curve with null tangent vector is zero.

Solution: the distance along a curve is

$$S = \int_C \sqrt{|g_{\mu\nu} t^\mu t^\nu|} d\lambda$$

where t^μ are the components of a tangent vector to the curve. For a null geodesic

$$\vec{t} \cdot \vec{t} = 0$$

$$\Rightarrow (t^\mu \vec{e}_\mu) \cdot (t^\nu \vec{e}_\nu) = t^\mu t^\nu \vec{e}_\mu \cdot \vec{e}_\nu = t^\mu t^\nu g_{\mu\nu} = 0$$

Substituting this into the expression above shows that the integrand vanishes if the curve is null. The distance along null geodesics is therefore zero.

Now let's differentiate $\vec{k} \cdot \vec{k} = 0$ to get

$$\frac{d\vec{k}}{d\lambda} \cdot \vec{k} + \vec{k} \cdot \frac{d\vec{k}}{d\lambda} = 2\vec{k} \cdot \frac{d\vec{k}}{d\lambda} = 0$$

$$\Rightarrow \frac{d\vec{k}}{d\lambda} = \beta \vec{k},$$

for some $\beta = f(\lambda)$. Recall that $\vec{k} = \frac{d\vec{s}}{d\lambda}$. This means that we can choose λ such that

$$\frac{d\vec{k}}{d\lambda} \equiv \nabla_{\vec{k}} \vec{k} = 0 \quad \Leftrightarrow \quad k^\mu D_\mu k^\nu = 0$$

For null curves, such a choice is called "affine parameter" (c.f. the affine parameter along timelike or spacelike curves, which was defined such that $\vec{t}\cdot\vec{t} = \pm 1$, and was called "proper time/distance").

Note that the 4-momentum of a photon can be simply taken as $\vec{k}$ (normally its $\vec{p} = m\vec{u}$ for particles with mass, but photons have no mass)

Example: prove that it is possible to choose λ such that $d\vec{k}/d\lambda = 0$.

Solution: recall that $\vec{k} = \frac{d\vec{x}}{d\lambda}$ and transform λ so that $\lambda \to f(\lambda)$

$$\Rightarrow \vec{k} = \frac{df}{d\lambda}\frac{d\vec{x}}{df} = \frac{df}{d\lambda}\vec{k}'$$

where $\vec{k}' \equiv \frac{d\vec{x}}{df}$. Now substitute into the expression

$$\frac{d\vec{k}}{d\lambda} = \nabla_{\vec{k}}\vec{k} = \beta\vec{k}$$

$$\Rightarrow \frac{d\vec{k}}{d\lambda} = \frac{df}{d\lambda}\nabla_{\vec{k}'}\left(\frac{df}{d\lambda}\vec{k}'\right) = \frac{df}{d\lambda}\left(\nabla_{\vec{k}'}\frac{df}{d\lambda}\right)\vec{k}' + \left(\frac{df}{d\lambda}\right)^2\nabla_{\vec{k}'}\vec{k}' = \beta\frac{df}{d\lambda}\vec{k}'$$

or

$$\frac{d^2 f}{d\lambda^2}\vec{k}' + \left(\frac{df}{d\lambda}\right)^2\nabla_{\vec{k}'}\vec{k}' = \beta\frac{df}{d\lambda}\vec{k}'$$

If we now choose $f(\lambda)$ such that

$$\frac{d^2 f}{d\lambda^2} = \beta \frac{df}{d\lambda}$$

then $\nabla_{\vec{k}'} \vec{k}' = 0$, as required.

2.6 Frequency of light

We've shown that light follows null geodesics. But how can we extract the frequency of a given photon? This must depend on the observer, because of the Doppler effect. Firstly, recall that in a given frame a photon's 4-momentum can be written

$$\vec{k} = k^a \hat{\vec{e}}_a = \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} \hat{\vec{e}}_a$$

$$\Rightarrow \vec{u} \cdot \vec{k} = k^a \vec{u} \cdot \hat{\vec{e}}_a = k^a \eta_{0a} = -E$$

and

$$\vec{x}_1 \cdot \vec{k} = k^a \vec{x}_1 \cdot \hat{\vec{e}}_a = k^a \eta_{1a} = p_x$$

$$\vec{x}_2 \cdot \vec{k} = p_y \qquad \text{and} \qquad \vec{x}_3 \cdot \vec{k} = p_z$$

These are the frame components of $\vec{k}$, for an observer following $\vec{u}$. We can use the Lorentz transformation rules to transform to a different frame, for a different observer. This gives the relativistic Doppler effect, between two observers in relative motion at a given point in spacetime. Explicitly for a boost in the x-direction,

$$E' = -k^{a'}\eta_{0'a'} = -L^{a'}{}_b k^b \eta_{0'a'} = L^{0'}{}_b k^b = L^{0'}{}_0 k^0 + L^{0'}{}_1 k^1 + L^{0'}{}_2 k^2 + L^{0'}{}_3 k^3$$

$$= \gamma k^0 + \gamma\beta k' = \gamma E + \gamma\beta p_x$$

Recall that for a photon $E = h\nu$ and $p = \frac{h}{\lambda} = h\nu$

$$\Rightarrow h\nu' = \gamma h\nu + \gamma\beta h\nu = (1+\beta)\gamma h\nu = \frac{(1+\beta)}{\sqrt{1-\beta^2}} h\nu$$

$$\Rightarrow \nu' = \frac{(1+\beta)}{\sqrt{1-\beta^2}}\nu = \frac{(1+\beta)}{\sqrt{(1+\beta)(1-\beta)}}\nu = \sqrt{\frac{1+\beta}{1-\beta}}\nu$$

This is the relativistic Doppler effect, and is identical to the expansion from SR. Note that this result is independent of coordinates. However, it is usually useful to use coordinates to find the quantities involved, once a geometry has been specified.

Example: Consider the Schwarzschild geometry

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\Omega^2$$

and a photon trajectory with coordinate components

$$k^\nu = A\left(\frac{1}{\left(1 - \frac{2GM}{r}\right)}, 1, 0, 0\right)$$

What frequency would an observer at fixed r measure for this photon?

Solution: first find u^μ:

$$\vec{u} \cdot \vec{u} = g_{\mu\nu} u^\mu u^\nu = -\left(1 - \frac{2GM}{r}\right)(u^0)^2 = -1$$

$$\Rightarrow u^\mu = \left(\frac{1}{\sqrt{1 - \frac{2GM}{r}}}, 0, 0, 0\right)$$

Now find E:

$$E = -\vec{u}\cdot\vec{k} = -g_{\nu\mu} u^\mu k^\nu = \left(1 - \frac{2GM}{r}\right) \frac{1}{\sqrt{1 - \frac{2GM}{r}}} \frac{A}{\left(1 - \frac{2GM}{r}\right)} = \frac{A}{\sqrt{1 - \frac{2GM}{r}}}$$

If two observers, at two fixed values of r, measure the same photon then the change in frequency is

$$\frac{\nu_2}{\nu_1} = \frac{E_2}{E_1} = \sqrt{\frac{1 - \frac{2GM}{r_1}}{1 - \frac{2GM}{r_2}}}$$

2.7 Stress-energy tensor

As well as light, we need a relativistic way to describe matter (i.e. energy density). It's clear that the density of matter cannot be a scalar because mass measured by two observers in relative motion is given by

$$m' = \gamma m_0$$

Likewise length contraction reduces the length of the edge of a box by $L' = L/\gamma$. These two factors together give, for the density ρ,

$$\rho' = \gamma^2 \rho_0$$

This suggest that ρ is the frame component of a rank-2 tensor. A simple version of such a tensor can be written

$$T = \rho \vec{u} \otimes \vec{u}$$

This is the simplest possible version of a stress-energy tensor. In the frame determined by $\vec{u}$:

$$T(\vec{u}, \vec{u}) = \rho(\vec{u} \cdot \vec{u})(\vec{u} \cdot \vec{u} = \rho(-1)(-1)) = \rho$$

while all other frame components of T vanish:

$$T(\vec{u}, \vec{x}_i) = 0 = T(\vec{x}_i, \vec{x}_j)$$

Now consider what happens in a boosted frame $\vec{e}'_a = L^b{}_{a'}\vec{e}_b$

$$\Rightarrow \vec{u}' = \gamma\vec{u} + \gamma v_i \vec{x}_i \qquad \text{and} \qquad \vec{x}'_i = \gamma v_i \vec{u} + \gamma \vec{x}_i$$

so

$$T(\vec{u}', \vec{u}') = \rho(\vec{u}'\vec{u})(\vec{u}' \cdot \vec{u}) = \rho\gamma^2(\vec{u} \cdot \vec{u})^2 = \rho\gamma^2 = \rho'$$

$$T(\vec{u}', \vec{x}'_i) = \rho(\vec{u}'\vec{u})(\vec{x}'_i \cdot \vec{u}) = \rho\gamma^2 v_i(\vec{u} \cdot \vec{u})^2 v_i = \rho\gamma^2 = \rho' v_i$$

$$T(\vec{x}'_i, \vec{x}'_i) = \rho(\vec{x}'_i\vec{u})(\vec{x}'_i \cdot \vec{u}) = \rho\gamma^2 v_i v_i(\vec{u} \cdot \vec{u})^2 v_i v_i = \rho\gamma^2 = \rho' v_i v_i$$

In this new frame we therefore have

$$T(\vec{u}', \vec{u}') \Leftrightarrow \text{energy density}$$

$$T(\vec{u}', \vec{x}'_i) \Leftrightarrow \text{energy flux density}$$

$$T(\vec{x}'_i, \vec{x}'_i) \Leftrightarrow \text{pressure, or stress}$$

This suggests a more general form for the stress-energy tensor:

$$T = T^{ab}\hat{\vec{e}}_a \otimes \hat{\vec{e}}_b = \begin{pmatrix} T^{00} & T^{0i} \\ T^{i0} & T^{ij} \end{pmatrix} \hat{\vec{e}}_a \otimes \hat{\vec{e}}_b = \begin{pmatrix} e_i & q^i \\ q^i & \pi^{ij} \end{pmatrix} \hat{\vec{e}}_a \otimes \hat{\vec{e}}_b$$

where

$$q^i \equiv -T(\vec{u}, \vec{x}^i)$$

$$\pi^{ij} \equiv T(\vec{x}^i, \vec{x}^j)$$

$$\rho \equiv T(\vec{u}, \vec{u})$$

If $q^i = 0$ and $\pi^{ij} = p\delta^{ij}$ then we say that T represents a "perfect fluid". In this case only ρ and p are non-zero. A relationship between p and ρ is known as an "equation of state"

Examples:

$$p = 0 \quad \text{is dust}$$

$$p = \frac{1}{3}\rho \quad \text{is radiation}$$

$$p = \rho \quad \text{is a stiff fluid}$$

For non relativistic fluids we usually have $q^i \ll \rho$ and $\pi^{ij} \ll \rho$.

Example: show that for a general matter distribution we can write

$$T^{\mu\nu} = \rho u^\mu u^\nu + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu} \,,$$

where $q^\mu u_\mu = 0 = \pi^{\mu\nu} u_\mu$.

Solution: using this expression, and the mutually orthogonal nature of the basis vectors, we find

$$\begin{aligned}
\vec{T}(\vec{u},\vec{u}) &= T^{\mu\nu} u_\mu u_\nu \\
&= \rho u^\mu u^\nu u_\mu u_\nu + q^\mu u^\nu u_\mu u_\nu + q^\nu u^\mu u_\mu u_\nu + \pi^{\mu\nu} u_\mu u_\nu \\
&= \rho(-1)^2 = \rho \\
\vec{T}(\vec{u},\vec{x}_i) &= T^{\mu\nu} u_\mu x_{i\nu} \\
&= \rho u^\mu u^\nu u_\mu x_{i\nu} + q^\mu u^\nu u_\mu x_{i\nu} + q^\nu u^\mu u_\mu x_{i\nu} + \pi^{\mu\nu} u_\mu x_{i\nu} \\
&= -q_i \\
\vec{T}(\vec{x}_i,\vec{x}_j) &= T^{\mu\nu} x_{i\mu} x_{j\nu} \\
&= \rho u^\mu u^\nu x_{i\mu} x_{j\nu} + q^\mu u^\nu x_{i\mu} x_{j\nu} + q^\nu u^\mu x_{i\mu} x_{j\nu} + \pi^{\mu\nu} x_{i\mu} x_{j\nu} \\
&= \pi_{ij}
\end{aligned}$$

The proposed $T^{\mu\nu}$ therefore gives all the correct frame components for the stress-energy tensor.

2.8 Stress-energy conservation

An important concept in Newtonian physics is energy conservation. This is encoded in the Newtonian equation of continuity:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

The relativistic (covariant) way of writing this equation is

$$D_\mu(\rho u^\mu) = 0$$

Now consider taking the divergence of $T^{\mu\nu} = \rho u^\mu u^\nu$

$$D_\mu T^{\mu\nu} = D_\mu(\rho u^\mu) u^\nu + \rho u^\mu D_\mu u^\nu$$

We therefore find that if stress-energy is conserved, then the particles in the fluid must be following geodesics:

$$D_\mu T^{\mu\nu} = 0 \qquad \Leftrightarrow \qquad u^\mu D_\mu u^\nu = 0$$

Note that this correspondence is only true for perfect fluids when $p = 0$. In general we expect $\boxed{D_\mu T^{\mu\nu} = 0}$ when there are no external forces on the matter being described by $T^{\mu\nu}$.

2.9 Inertial frames

So far, we have only considered frame vectors at a single point in space-time. If we want to determine what an observer would measure at different points along their world line, however, we need to know something about to relate frame vectors at different points (i.e. we need to know how the frame is moving).

In general, a frame can be rotating (if it corresponds to a lab on the surface of the Earth, for example). It is often useful to use non-rotating frames though, to avoid coriolis and centrifugal forces.

A frame may also be accelerating if it's acted on by an external force (again, the surface of the Earth is a good example of this). In this case the matter making up the lab obeys

$$D_\mu T^{\mu\nu} \neq 0 \qquad \Leftrightarrow \qquad a^\mu \equiv u^\nu D_\nu u^\mu \neq 0$$

If a frame is non-rotating and accelerating at the rate a^μ then it is said to be "Fermi-propagated".

A Fermi propagated frame obeys the following equation

$$\frac{d\hat{\vec{e}}_a}{d\tau} = (\vec{u} \cdot \hat{\vec{e}}_a)\vec{a} - (\vec{a} \cdot \hat{\vec{e}}_a)\vec{u}$$

where the intrinsic derivative is along the observers worldline, and where τ is the observer's proper time.

If the frame is both non-rotating and non-accelerating then we say it is an "inertial frame". In this case $\vec{a} = 0$, so

$$\frac{d\vec{u}}{d\tau} = 0 \qquad \text{and} \qquad \frac{d\vec{x}_i}{d\tau} = 0$$

This means that inertial frames are parallel transported along the worldines of observers. They are often used in GR to calculate observables.

Note: non-inertial, accelerating frames contain fictitious forces (Coriolis, centrifugal etc.). In GR, the acceleration we feel on the surface of the Earth should be considered equally fictitious, as it is the result of an external force from the solid Earth that pushes off what be otherwise be a free fall trajectory.

2.10 Riemann curvature tensor

The curvature of spacetime is of fundamental importance in GR and can be quantified precisely by the Riemann curvature tensor. In a coordinate basis, the components of this tensor are

$$R^{\rho}{}_{\mu\nu\sigma} = \partial_{\mu}\Gamma^{\rho}{}_{\mu\sigma} - \partial_{\sigma}\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\tau}{}_{\mu\sigma}\Gamma^{\rho}{}_{\tau\nu} - \Gamma^{\tau}{}_{\mu\nu}\Gamma^{\rho}{}_{\tau\sigma}$$

To find $R^{\rho}{}_{\mu\nu\sigma}$ we can take $g_{\mu\nu}$, and use it to calculate $\Gamma^{\rho}{}_{\mu\nu}$, which is then substituted into the equation above. We can verify $R^{\rho}{}_{\mu\nu\sigma}$ are the components of a tensor by checking how it transforms under a change or coordinates.

The Riemann tensor obeys a number of important identities; for which it is sometimes useful to write $R_{\mu\nu\sigma\rho} = g_{\mu\tau}R^{\tau}{}_{\mu\sigma\rho}$. These are:

Ricci identities: for any vector with components v_{μ}

$$D_{\mu}D_{\nu}v_{\sigma} - D_{\nu}D_{\mu}v_{\sigma} = R'^{\tau}{}_{\sigma\nu\mu}v_{\tau}$$

First Bianchi identities:

$$R_{\mu\nu\sigma\rho} + R_{\mu\sigma\rho\nu} + R_{\mu\rho\nu\sigma} = 0$$

Second Bianchi identities:

$$D_\tau R_{\mu\nu\sigma\rho} + D_\sigma R_{\mu\nu\rho\tau} + D_\rho R_{\mu\nu\tau\sigma} = 0$$

Skew symmetries:

$$R_{\mu\nu\sigma\rho} = -R_{\nu\mu\sigma\rho} = -R_{\mu\nu\rho\sigma}$$

Interchange symmetry:

$$R_{\mu\nu\sigma\rho} = R_{\sigma\rho\mu\nu}$$

These properties restrict the number of independent components of $R^\mu{}_{\nu\rho\sigma}$ to 20.

Note that spacetime is flat, and SR is recovered, if and only if all components of the Riemann tensor vanish at every point. Any non-zero components of the Riemann tensor mean that spacetime is not flat

2.11 Ricci and Weyl tensors

The Riemann tensor contains a lot of information. It is useful to break it up into smaller parts: The Ricci tensor $R_{\mu\nu}$ and the Weyl tensor $C_{\tau\mu\sigma\rho}$.

The Ricci tensor is given in a coordinate basis by the following components:

$$R_{\mu\nu} \equiv R^{\sigma}{}_{\mu\sigma\nu}$$

This tensor has 10 independent components, and can be thought of (in some sense) as the trace of the Riemann tensor.

The Weyl tensor has components, in a coordinate basis, as follows:

$$C_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma} - \frac{1}{2} g_{\mu\rho} R_{\nu\sigma} + \frac{1}{2} g_{\mu\sigma} R_{\nu\rho} + \frac{1}{2} g_{nu\rho} R_{\sigma\mu} - \frac{1}{2} g_{\nu\sigma} R_{\rho\mu} + \frac{1}{6} R g_{\mu\rho} g_{\sigma\nu} - \frac{1}{6} R g_{\mu\sigma} g_{\rho\nu}$$

where $R \equiv R^{\mu}{}_{\mu}$ is the Ricci scalar. The Weyl tensor has 10 independent components and can be thought of as the trace-free part of the Riemann tensor (the contraction over any two indices yields zero, $C^{\mu}{}_{\nu\mu\rho} = 0$).

Together, $R_{\mu\nu}$ and $C_{\mu\nu\rho\sigma}$ contains all the information in $R^{\mu}{}_{\nu\rho\sigma}$.

2.12 Einstein's equations

Einstein proposed the following set of equations to describe how matter matter curves space-times:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} - g_{\mu\nu}\Lambda$$

or equivalently,

$$\boxed{R_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) + g_{\mu\nu}\Lambda}$$

These equations show that if we know the value of the stress-energy at any point in space-times then we automatically know the Ricci curvature at that point.

Note: Einstein's equations only provide enough information to specify half of the Riemann tensor - the Weyl tensor remains completely unspecified, even if we know everything about the distribution of the matter (and hence $R_{\mu\nu}$). In practise, the Weyl tensor components are determined by boundary conditions, or by imposing specific properties on the spacetime.

The value of the constant Λ, in these equations, is not specified by the theory either. It must be determined observationally, and can only take one value for the entire universe.

2.13 Empty space I: Minkowski

If the space is empty ($T_{\mu\nu} = 0$) and the cosmological constant vanishes ($\Lambda = 0$) then Einstein's equations imply

$$R_{\mu\nu} = 0$$

This means that any metric that has zero Ricci curvature is an empty space solution. There are very many such metrics.

The simplest example is Minkowski space, which can be written as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

This solution has $C_{\mu\nu\rho\sigma} = 0$ and $R^{\mu}{}_{\nu\rho\sigma} = 0$. All coordinates run from $-\infty$ to $+\infty$, so we can say the spacetime is infinitely extended. Nevertheless, there is a simple finite diagram that we can draw in order to understand this space.

First consider the space spanned by the t and x coordinates:

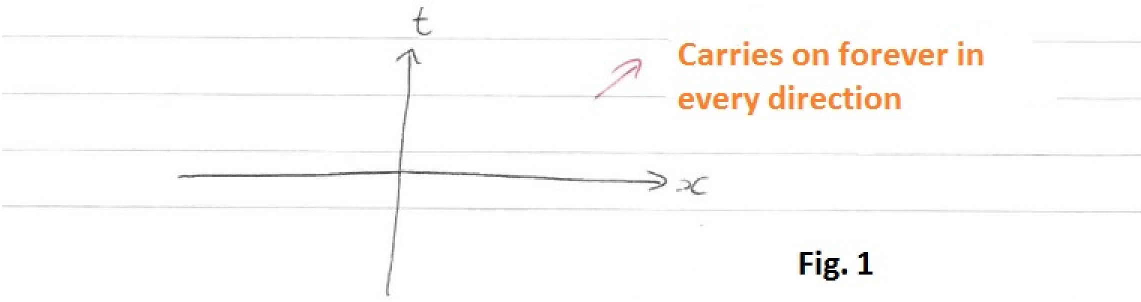

Fig. 1

Rays of light, in this diagram, would be represented by diagonal lines (assuming they only propagate in the x-direction). Now, consider shrinking this diagram in every direction, so that light rays stay diagonal, If we shrink the diagram sufficiently, we can "compactify" it to a finite size. It then looks like this:

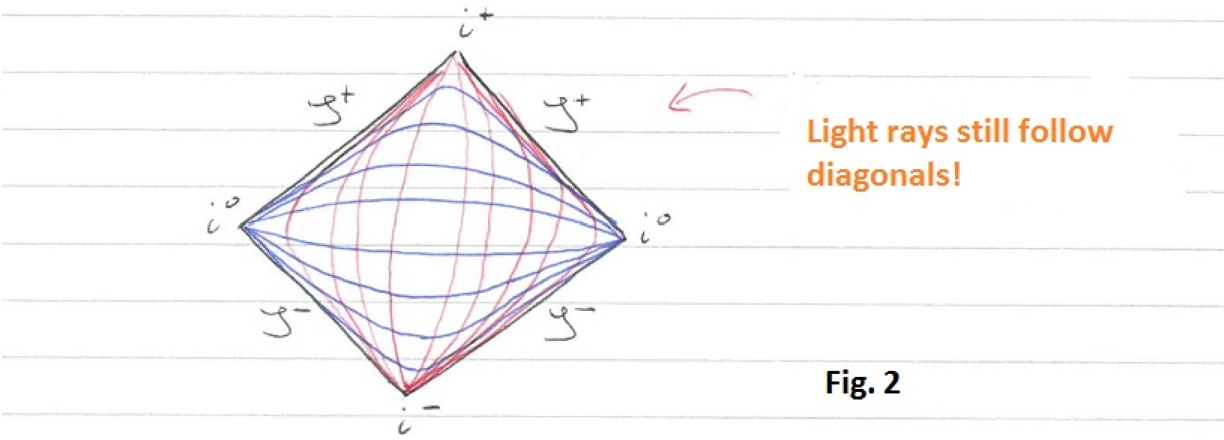

Fig. 2

In this diagram:

- Blues lines are surfaces of constant t.
- Red lines are lines of constant x.

- All timelike particles end up at i^+
- All timelike particles started at i^-
- All light rays of light end up at $\Im^+$
- All light rays started at $\Im^-$
- All spacelike curves end up at i^0

This is an example of a "Penrose diagram"

2.14 Empty space II: Milne

In GR it's not always clear if two different line elements represent two different space times, or if they are just the same spacetime written in different coordinates. to illustrate this consider again Minkowski (this time using spherical polar coordinates):

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Now introduce new coordinates

$$r = RT \qquad \text{and} \qquad t^2 = T^2(1 + R^2)$$

$$dr = RdT + TdR$$

therefore

$$dr^2 = R^2 dT^2 + T^2 dR^2 + 2TRdTdR$$

and

$$2tdt = 2T(1 + R^2)dT = 2T^2 RdR$$

so

$$dt^2 = T^2(1 + R^2)dT^2 + T^4 R^2 dR^2 + 2T^3(1 + R^2)dRdT$$

$$= (1 + R^2)dT^2 + \frac{T^2R^2}{(1 + R^2)} dR^2 + 2TRdTdR$$

which implies

$$-dt^2+dr^2 = -(1+R^2)dT^2 - \frac{T^2R^2}{(1 + R^2)} dR^2 - 2TRdTdR + R^2dT^2 + T^2dR^2 + 2TRdTdR$$

so

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\Rightarrow ds^2 = -dT^2 = a^2(T)\Big(\frac{dR^2}{1 - kR^2} + R^2(d\theta^2 + \sin^2\theta d\phi^2)\Big)$$

where $a(T) = T$ and $k = -1$, and where T and R run from 0 to ∞. This looks like an expanding universe!

On a Penrose diagram:

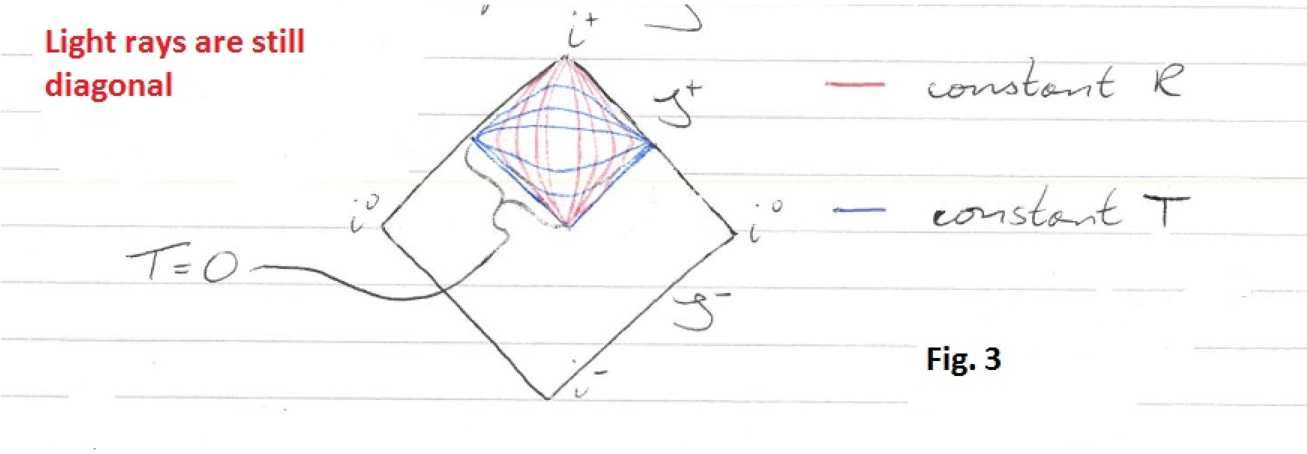

Fig. 3

Note: the new coordinates only cover a quarter of the spacetime! Note also that we still have $R_{\mu\nu} = 0 = C_{\mu\nu\rho\sigma}$, which should have been enough to inform us that Minkowski space and Milne space are identical (up to a change of coordinates).

2.15 Empty space III: Plane waves

Not all vacuum spacetimes are related to Minkowski space by coordinate transformations. For example, consider

$$ds^2 = -dt^2 + dx^2 + p^2(u)dy^2 + q^2(u)dz^2$$

where $u \equiv t - x$. The functions $p(u)$ and $q(u)$ obey wave equations:

$$\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = 0$$

This geometry is a solution to Einstein's equations if

$$\frac{\ddot{p}}{p} + \frac{\ddot{q}}{q} = 0 \Leftrightarrow R_{\mu\nu} = 0$$

However, they have non-zero Weyl curvature if

$$\ddot{p} \neq 0 \neq \ddot{q} \qquad \Leftrightarrow \qquad C_{\mu\nu\rho\sigma} \neq 0$$

If a region of spacetime has $C_{\mu\nu\rho\sigma} \neq 0$ then it can never be transformed to Minkowski space. This is a (very special) example of a gravitational wave. We will return to GWs later on in this course.

2.16 Empty space IV: de Sitter space

So far we've only considered $\Lambda = 0$. If we allow $\Lambda \neq 0$ then we have even more geometries that satisfy Einstein's equations. The most symmetric of these is the de Sitter space

$$ds^2 = -dt^2 + \frac{3}{\Lambda} \cosh^2 \left(\sqrt{\frac{\Lambda}{3}} t \right) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)$$

This can be seen to reduce to Minkowski space (in spherical coords) when $\Lambda \to 0$. The properties of de Sitter space are, however, very different to Minkowski.

Consider the Penrose diagram:

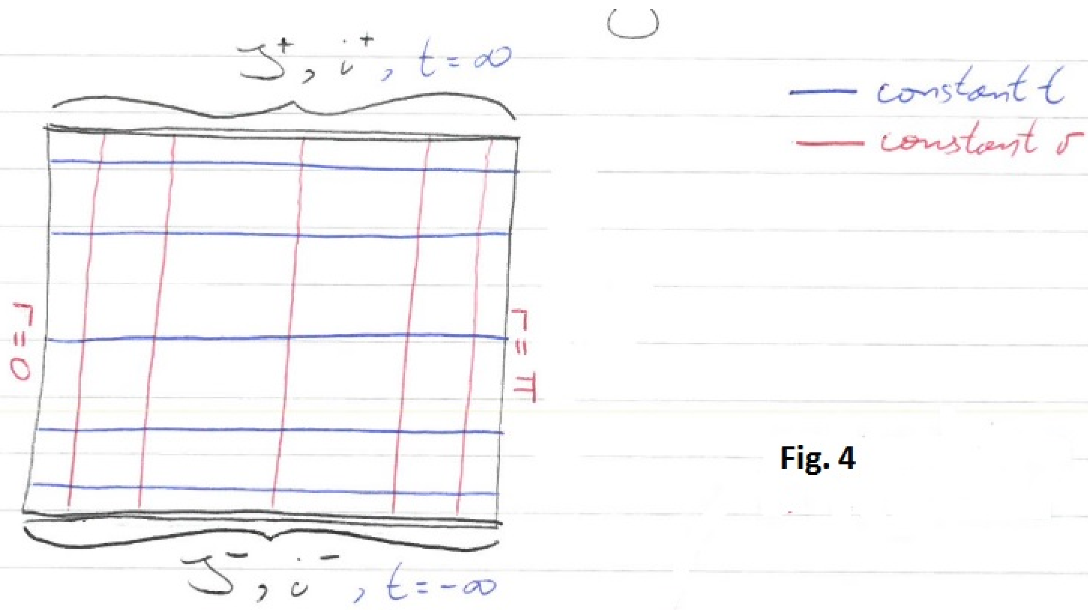

Fig. 4

All null and timelike infinities are now spacelike surfaces! This geometry can now be transformed using

$$T \equiv \sqrt{\frac{3}{\Lambda}} \tanh^{-1}\left(\frac{\tanh\left(\sqrt{\frac{\Lambda}{t}}t\right)}{\cos r}\right), \quad R \equiv \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}}t\right) \sin r$$

which gives

$$\Rightarrow ds^2 = -(1 - \frac{\Lambda}{3}R^2)dT^2 + \frac{dR^2}{(1 - \frac{\Lambda}{3}R^2)} + R^2(d\theta^2 + \sin^2\theta d\phi^2)$$

This looks a bit like Schwarzchild!

These coordinates don't cover the whole Penrose diagram:

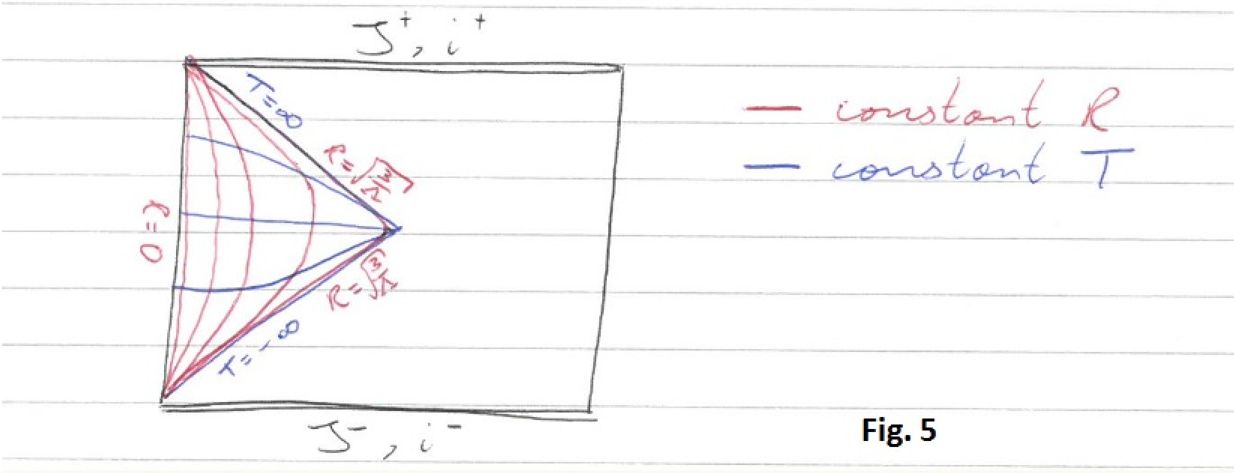

Fig. 5

Note: the Penrose diagram itself (the black lines in the figure) is a consequence of the causal structure of the spacetime, and is therefore unaffected by the change in coordinates.

Going even further, we can define new coordinates ρ and τ via

$$\tau \equiv T + \sqrt{\frac{3}{\Lambda}} \ln \sqrt{\frac{3}{\Lambda} - R^2}, \quad \rho \equiv R e^{-\sqrt{\frac{\Lambda}{3}}\tau}$$

The same spacetime now looks like an exponentially expanding universe:

$$\Rightarrow ds^2 = -d\tau^2 + e^{2\sqrt{\frac{\Lambda}{3}}\tau}(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2)$$

Again, these new coordinates do not cover the entire spacetime:

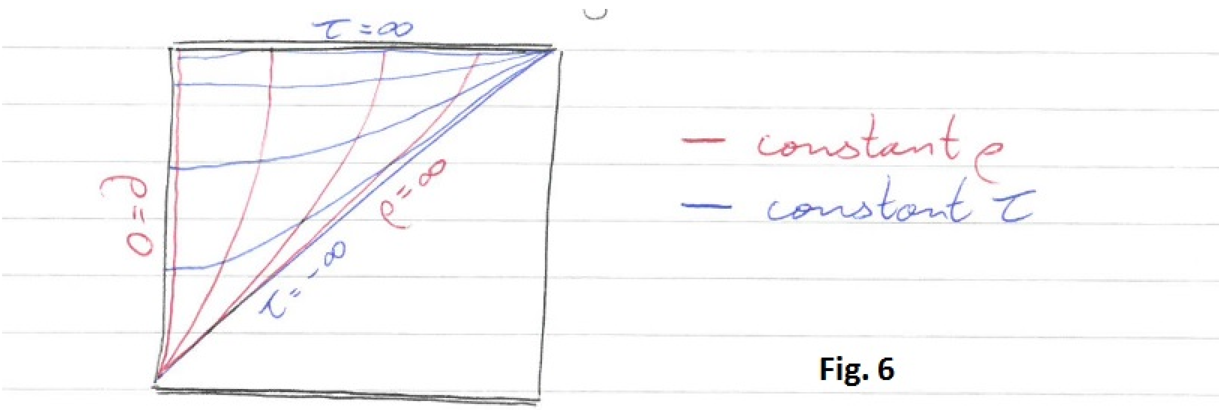

Fig. 6

Note: all of the descriptions of de Sitter above are Ricci curved ($R_{\mu\nu} \neq 0$) and Weyl flat $C_{\mu\nu\rho\sigma} = 0$. Coordinate transformations cannot change these properties.

2.17 One-body solution I: Schwarzschild

In spacetime and gravity you have already come across Schwarzschild's solution

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\theta^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Birkhoff's theorem: When $\Lambda = 0$, the unique vacuum spherically symmetric solution is given by Schwarzchild.

This means:

- no spacetime can be dynamical if it is empty and spherically symmetric

- there can be no spherically symmetric gravitational waves.

- the exterior geometry of spacetime around every spherically symmetric body (e.g. stars) must be identical, and is given by Schwarzschild

The Penrose diagram for Schwarzchild:

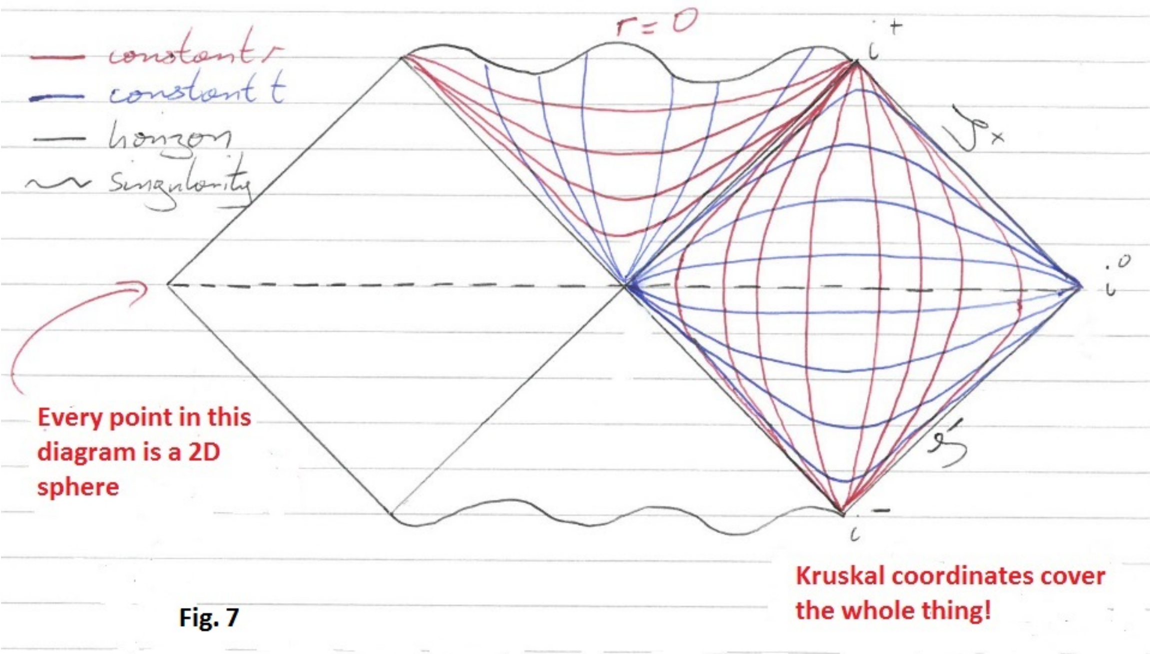

Fig. 7

Note:

- the r and t coordinates can not be used to label points on the horizon

- the black hole singularity is in the future

- there is another singularity in the past (a "white hole")

- the black hole has two exterior regions which are causally disconnected

- the black dashed time contains a wormhole at $r = 2Gm$:

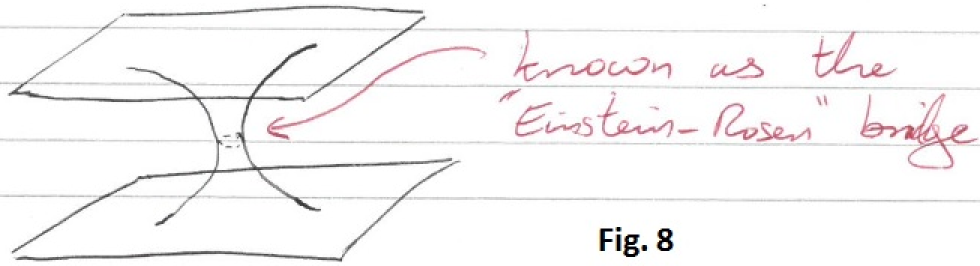

Fig. 8

Note that Schwarzschild has $R_{\mu\nu} = 0$ and $C_{\mu\nu\rho\sigma} \neq 0$.

2.18 One-body solution II: Kerr

If we consider non-spherical geometries, then there are many other possible solutions. One of the most interesting is Kerr:

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\big(adt - (r^2 + a^2)d\phi\big)$$

where

$$\rho^2 \equiv r^2 + a^2\cos^2\theta$$

$$\Delta \equiv r^2 - 2GMr + a^2$$

and where m and a are constants. The Kerr solution is:

- axially symmetric ($g_{\mu\nu}$ is not a function of ϕ)

- stationary ($g_{\mu\nu}$ is not a function of t)

- vacuum ($R_{\mu\nu} = 0$)
- asymptotically flat $g_{\mu\nu} \to \eta_{\mu\nu}$ as $r \to \infty$
- contains horizons (i.e. is a black hole)

A result from Carter states that the Kerr geometry is the only solution of Einstein's equations that obeys these five conditions.

The Kerr geometry represents a rotating black hole (but not a rotating star!). Kerr contains singularities and horizons but they are very different from Schwarzchild. To find the location of the singularities we can evaluate the following scalar:

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48m^2}{(r^2 + a^2\cos^2\theta)}(r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^4\theta - a^6\cos^6\theta)$$

This diverges at $r = 0$ and $\theta = \frac{\pi}{2}$, which corresponds to a ring:

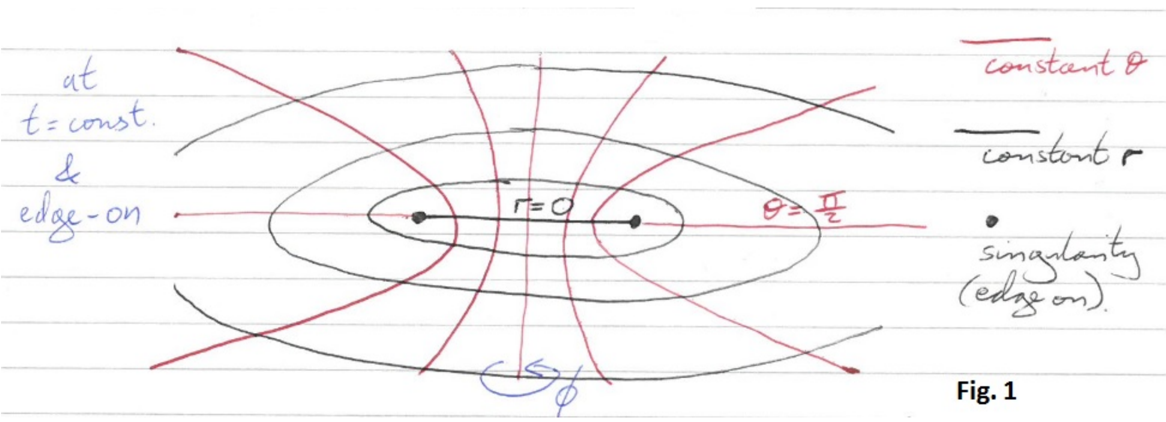

Fig. 1

The points labelled $r = 0$ are actually a disc (viewed edge on, above).

If $r = 0$ and $\theta \neq \frac{\pi}{2}$ then there is no singularity at that point, and we can continue the spacetime to a region with $r < 0$ - this corresponds to a region with negative mass! It is also a region that contains closed timelike curves, meaning that observers who follow them can travel into their own past!

Thankfully, all this strangeness is hidden behind the horizons. The horizons of the Kerr geometry are located at the coordinate singularities:

$$\Delta = r^2 - 2GMr + a^2 = 0$$

$$\Rightarrow r = r^{\pm}_{horizon} \equiv GM \pm \sqrt{G^2M^2 - a^2}$$

The two branches correspond to two distinct horizons - an inner one and an outer one.

Note: the rotation parameter can only take values such that $a^2 < G^2m^2$, otherwise there are no horizons.

Now consider Kerr's Penrose diagram:

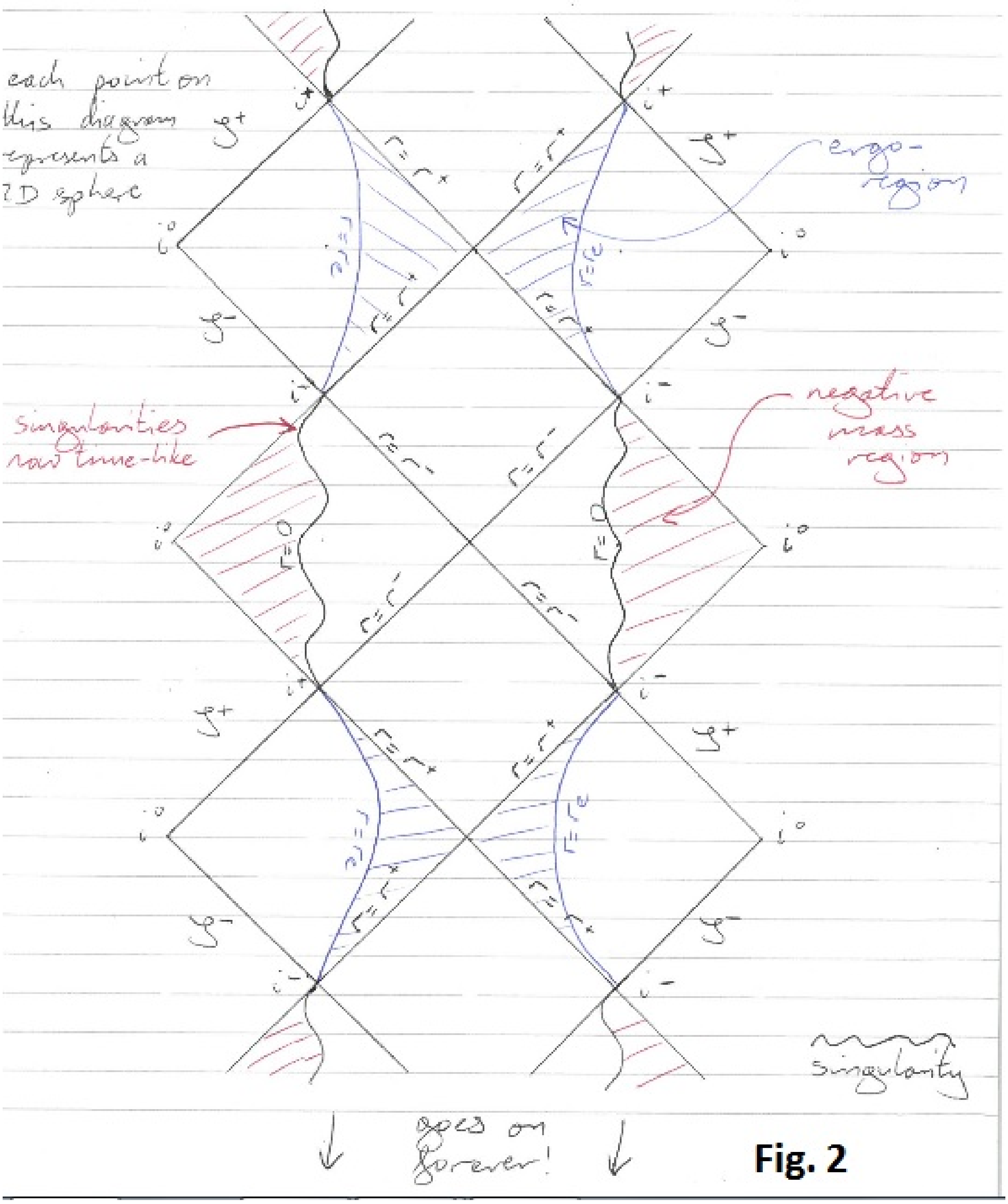

Fig. 2

Kerr contains a region of space, outside the outer horizon, within which massive particles cannot stay at fixed position with respect to distant stars: they are forced to rotate with the black hole, as the black hole pulls space around with it. This region is called the "ergoregion".

Exercise: show that the ergoregion of Kerr is bounded by

$$r \equiv GM + \sqrt{G^2M^2 - a^2\cos^2\theta}$$

Solution: if the observer stays at fixed (r, θ, ϕ)

$$\vec{u} = u^t \vec{e}_t$$

$$\Rightarrow \vec{u}\cdot\vec{u} = g_{tt}(u^t)^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\rho^2}(u^t)^2 = -1$$

$$\Rightarrow u^t = \frac{\rho}{\sqrt{\Delta - a^2\sin^2\theta}}$$

The timelike condition $\vec{u}\cdot\vec{u} < 0$ is only satisfied if u^t is real. This requires

$$\Delta > a^2\sin^2\theta \Leftrightarrow r^2 - 2GMr + a^2\cos^2\theta > 0$$

This last inequality is satisfied if $r > r_e$ where $r_e = GM + \sqrt{G^2M^2 - a^2\cos^2\theta}$, which must be the boundary of the ergoregion.

note: if $\theta = 0$ or π then $r_e = r^+$, so the ergosphere and horizon touch. If $0 < \theta < \pi$ then $\cos^2 < 1$, so $G^2M^2 - a^2\cos^2 > G^2M^2 - a^2$. This means $r_e > r^+$, so the ergosphere is outside the horizon. This is shown in the diagram below:

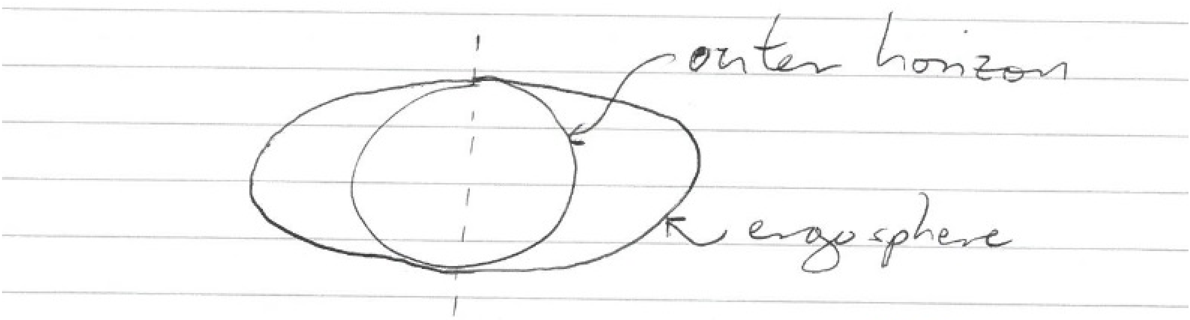

Fig. 3

2.19 One-body solution III: General case

The exterior region of Kerr ($r > r^+$) is thought to be the generic end state of gravitational collapse to a black hole. This is supported by computer simulations, but has yet to be proven (rigorously) in mathematics.

Astrophysical black holes are expected to be close to the limit $a^2 \approx G^2M^2$, because conservation of angular momentum suggests rotation increases as collapse occurs (note: it is impossible to spin a black hole up beyond $a^2 = G^2M^2$, if it starts below this value initially).

Kerr, while very interesting, can only ever be an approximation to real black holes. In astrophysics, black holes always have accretion discs (i.e.

are not perfect vacuum). They also do not exist in isolation (i.e. there are other objects in the universe). It is known that the interior of Kerr ($r < r^+$) is unstable to perturbations.

2.20 Multi-body solutions

Unlike in Newtonian physics, the two body problem has not yet been solved in GR. However, in the last 15 years we have reached the stage where we can simulate this problem on a computer - an area known as "Numerical relativity". This is an exciting field of research, and has been used to shed light on the many different ways that two BHs can orbit each other and merge.

Aside from advanced computer simulations, we can use perturbative techniques to try and understand the physics at work in multi-body systems in GR. The most advanced of these is the "effective one-body formalism", which is capable of calculating the gravitational wave signal from two merging black holes. This builds on the "post-Newtonian" formalism that describes weak gravitational fields, and the "Regge-Wheeler equations" that describe small pertubations to exact BH solutions. Multi-body solutions are essential for understanding gravitational waves signals.