Mathematical Methods (Second Year) MT 2009

Problem Set 5: Partial Differential Equations

1. *'Linear' and 'Homogeneous'*

   What is meant by the terms (a) linear and (b) homogeneous as applied to differential equations? Give a physical example of a second order linear non-homogeneous ordinary differential equation, and of a second order linear homogeneous partial differential equation. In both examples, discuss whether the linearity is believed to be exact, or is the result of an approximation (and if so, say what it is). In the case of a second order partial differential equation, describe the properties of the solutions that follow when it is (a) linear (b) linear and homogeneous.

2. *"Method of Characteristics"*

   A function $u(r,t)$ satisfies the equation

   $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

   where c is a constant. By introducing the new function $v(r,t) = ru(r,t)$, and writing $\xi = r + ct, \eta = r - ct$, reduce this equation to

   $$\frac{\partial^2 v}{\partial \xi \partial \eta} = 0.$$

   Hence show that the general solution $u(r,t)$ has the form $u(r,t) = \frac{1}{r}[f(r+ct)+g(r-ct)]$, where f and g are arbitrary (twice-differentiable) functions. What does the solution represent physically if t is the time and r is the 3-D polar variable?

3. *Modes of a 1-D string*

   The transverse displacement $y(x,t)$ (assumed small) of a string stretched between the points $x = 0$ and $x = a$ satisfies the equation

   $$\frac{\partial^2 y}{c^2 \partial t^2} = \frac{\partial^2 y}{\partial x^2}.$$

   Find the solution satisfying each of the following initial ($t = 0$) conditions:-

   (i) $y(x,0) = L\sin(\pi x/a)$, $\frac{\partial y}{\partial t}(x,0) = 0$;

   (ii) $y(x,0) = 0$, $\frac{\partial y}{\partial t}(x,0) = V\sin(2\pi x/a)$;

   (iii) $y(x,0) = L\sin(\pi x/a)$, $\frac{\partial y}{\partial t}(x,0) = V\sin(2\pi x/a)$;

   (iv) $y(x,0) = 2L\sin(\frac{3\pi x}{2a})\cos(\frac{\pi x}{2a})$, $\frac{\partial y}{\partial t}(x,0) = 0$.

   Sketch the displacement of the string in part (iv) at time $t = a/2c$.

4. *Damped thermal wave*

   The temperature T in a one-dimensional bar whose sides are perfectly insulated obeys the heat flow equation

   $$\frac{\partial T(x,t)}{\partial t} = K\frac{\partial^2 T(x,t)}{\partial x^2}$$

   where K is a constant. The bar extends from $x = 0$ to $x = \infty$. The temperature at the end $x = 0$ oscillates in time according to $T(x = 0, t) = T_0 \cos \omega t$. By looking for solutions that are separated in x and t ($T = X(x)F(t)$) find the solution for all $x \geq 0$, and t, which matches the boundary condition at $x = 0$. Sketch T versus x for $\omega t = \pi/2$, given that $\frac{\omega}{2K} = \frac{1}{a^2}$.

5. *2-D Laplace equation in polars*

   Laplace's equation in two dimensions may be written, using plane polar coordinates r, θ, as

$$r\frac{\partial}{\partial r}\left(r\frac{\partial V(r,\theta)}{\partial r}\right) + \frac{\partial^2 V(r,\theta)}{\partial \theta^2} = 0.$$

   Find all separable solutions of this equation which have the form $V(r,\theta) = R(r)S(\theta)$, which are single valued for all r, θ. What property of the equation makes any linear combination of such solutions also a solution?

   A continuous potential $V(r,\theta)$ satisfies Laplace's equation everywhere except on the concentric circles $r = a, r = b$ where $b > a$.

   (i) Given that $V(r = a, \theta) = V_0(1 + \cos\theta)$, and that V is finite as $r \to 0$, find V in the region $r \leq a$.

   (ii) Given, separately, that $V(r = b, \theta) = 2V_0 \sin^2\theta$, and V is finite as $r \to \infty$, find V for $r \geq b$.

6. *Modes of a square membrane*

   The equation governing the motion of a square vibrating membrane has the form

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

   with $u = 0$ on the boundary.

   Show that the normal modes are given by

$$u(x, y, t) = A_{m,n}\sin(m\pi x/L)\sin(n\pi y/L)\cos(\omega_{m,n}t)$$

   where $\omega_{m,n}$ is the frequency of the mode and L is the length of a side of the square. Express $\omega_{m,n}$ in terms of m and n and give any restrictions which apply to m and n.

   Show that the second lowest frequency is a factor of $\sqrt{\frac{5}{2}}$ larger than the frequency of the lowest mode, that two modes have this frequency and that by combining them it is possible to have a node along either diagonal of the square. Deduce the fundamental mode for a triangular membrane obtained by constraining the membrane along a diagonal.

7. *Cylindrical symmetry* $\to$ *Bessel functions*

   A resonant electromagnetic cavity is in the shape of a right circular cylinder of radius a and height h. Taking the z-axis along the axis of the cylinder, the electric field in this direction satisfies

$$\nabla^2 E_z + \frac{\omega^2}{c^2}E_z = 0$$

   for oscillatory solutions. When this is separated in cylindrical polar coordinates the solution has the form $R(\rho)\Phi(\phi)Z(z)$ where

$$\frac{d^2 Z}{dz^2} = -p^2 Z, \frac{d^2\Phi}{d\phi^2} = -m^2\Phi, \rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + [(\frac{\omega^2}{c^2} - p^2)\rho^2 - m^2]R = 0.$$

   Write $\omega^2/c^2 - p^2 = \gamma^2$ and introduce $x = \gamma\rho$. Show that the R equation is equivalent to

$$x\frac{d}{dx}\left(x\frac{dJ_m(x)}{dx}\right) + (x^2 - m^2)J_m(x) = 0.$$

   The boundary conditions on $z = 0, h$ require $Z \sim \cos(n\pi z/h)$ and that $R(\rho = a) = 0$. Show that the frequencies of allowed modes are given by

$$\frac{\omega^2}{c^2} = \frac{\alpha_{mr}^2}{a^2} + \frac{n^2\pi^2}{h^2}$$

where α_{mr} is the r^{th} root of J_m, i.e. $J_m(\alpha_{mr}) = 0$.

8. *Spherical symmetry* $\rightarrow$ *Spherical Harmonics*

   When Laplace's equation $\nabla^2 V = 0$ is separated in spherical polar coordinates, the solutions have the form $V = R(r)T(\theta)P(\phi)$ where

$$\frac{d^2P}{d\phi^2} = -m^2P, -\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dT}{d\theta} + \frac{m^2}{\sin^2\theta}T = \lambda T.$$

   (i) Why is m an integer or zero? (ii) Verify that for $m = 0, T = \cos\theta$ is a solution for a certain value of λ, and find that value. (iii) Show that the equation for R is

$$\frac{d}{dr}(r^2\frac{dR}{dr}) = \lambda R.$$

   Find the solutions for R corresponding to the particular value of λ in (ii). (iv) What is the most general solution for V having this θ, ϕ dependence? (v) An uncharged conducting sphere of radius a is placed at the origin in an intially uniform electrostatic field (0,0,E). Show that it behaves as an electric dipole. [Hint: V satisfies the boundary condition $V(r = a, \theta) = 0$ for all θ on $r = a$, since it's a conductor. The other condition required to fix V uniquely is the one at $r \rightarrow \infty$. At very large distances from the sphere, the field must be essentially the same as in the absence of the field, giving $V(r \rightarrow \infty) = -Ez = -Er\cos\theta$. Use these two boundary conditions to determine the arbitary constants introduced in (iv)].

9. The temperature T is a one-dimensional bar whose sides are perfectly insulated obeys the heat flow equation

$$\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial x^2}$$

   where κ is a positive constant. The bar extends from $x = 0$ to $x = L$ and is perfectly insulated at $x = L$. At $t < 0$ the temperature is 0^0 C throughout the bar and at $t = 0$ the uninsulated end is placed in contact with a heat bath at 100^0C. Show that the temperature of the bar at subsequent times is given by

$$\frac{T}{100} = 1 - \sum_{n=0}^{\infty}\frac{4}{(2n+1)\pi}\sin\left(\frac{(2n+1)\pi x}{2L}\right)\exp\left\{-\kappa\left(\frac{(2n+1)\pi}{2L}\right)^2 t\right\}.$$

   Estimate the time taken for the average temperature of the bar to reach 90^0 C if $L = 1$m and $\kappa = 0.001\text{m}^2\text{s}^{-1}$.

10. *Use of other eigenfunction equations*

    Laplace's equation in spherical polar coordinates (r, θ, ϕ) for a potential $V(r, \theta)$ which is independent of ϕ is

$$\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) = 0.$$

    (i) Show that there is a solution of the form

$$V(r, \theta) = \sum_{n=0}^{\infty} P_n(\cos\theta)(\frac{\alpha_n}{r^{n+1}} + \beta_n r^n)$$

where $P_n(x)$ satisfies

$$\frac{d}{dx}\left((1-x^2)\frac{dP_n(x)}{dx}\right) + n(n+1)P_n(x) = 0.$$

(ii) By first calculating the potential along the z-axis, find the potential at arbitrary θ and $r > a$ for the following charge distributions:

a) A dipole lying along the z-axis and consisting of a charge $+1$ at $z = +a$ and a charge -1 at $z = -a$.

b) A quadrupole lying along the z-axis and consisting of a charge -1 at $z = +a$, a charge $+2$ at $z = 0$, and a charge -1 at $z = -a$.

You may assume $P_n(1) = 1$.