# Imperial College London
# BSc/MSci EXAMINATION June 2018

# MPH2 MATHEMATICAL METHODS

## For 2nd, 3rd and 4th year Physics students

Monday, 4 June, 2018: 14:00 to 16:00

*Answer question 1 and two of questions 2, 3, 4 and 5. A mathematical formula sheet is provided.*
*Marks shown on this paper are indicative of those the Examiners anticipate assigning.*

**General Instructions**

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Complete the front cover of each of the 3 answer books provided, entering the number of the question attempted in the box on the front cover of the corresponding answer book.

Hand in 3 answer books even if they have not all been used.

Examiners attach great importance to legibility, accuracy and clarity of expression.

$$\text{Fourier transform} \quad \hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$\text{Fourier integral} \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk.$$

**1.** (i) Write down Cauchy's integral formula and state *briefly* the conditions under which the formula holds.

(ii) Consider the integral

$$\int_a^b \frac{x\, y'(x)^2}{2}\, dx,$$

where $b > a > 0$. Solve the associated Euler-Lagrange equation.

(iii) Let

$$v(x, y) = \frac{y}{x^2 + y^2}.$$

Find an analytic function $f$ with imaginary part $v$.

(iv) Consider the function

$$f(z) = \frac{1}{2 - z}.$$

(a) Write down the Taylor series expansion of $f(z)$ about $z = 0$.

(b) Obtain the Laurent series expansion of $f(z)$ valid in the annulus $|z| > 2$.

(v) Simplify (a) $\epsilon_{ipq}\epsilon_{jpq}$ and (b) $\epsilon_{ijk}\epsilon_{ijk}$.

(vi) The Fourier transform of

$$f(x) = \frac{1}{1 + x^2} \quad \text{is} \quad \hat{f}(k) = \frac{e^{-|k|}}{2}.$$

Obtain the Fourier transform of

$$g(x) = \frac{\cos x}{1 + x^2}.$$

Hint: use $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$.

(vii) Find in the form of a Fourier integral a solution of the ODE

$$\ddot{x}(t) + \gamma\dot{x}(t) = \frac{1}{1 + t^2} - \pi\delta(t),$$

where $\gamma$ is a positive constant (your solution need not be real).

(viii) Define what is meant by polar and axial vectors. What kind of vector is the cross product of a polar and an axial vector? Briefly justify your answer.

[Total 40 marks]

**2.** (i) The motion of a physical system with one generalised coordinate $q$ is described by the Lagrangian $L(q, \dot{q})$ which has no explicit time-dependence. Show that

$$H = \dot{q}\frac{\partial L}{\partial \dot{q}} - L$$

is a constant of the motion.

(ii) A bead of mass $m$ moves without friction or gravity on a heart-shaped wire described in polar coordinates by the equation $r = 1+\cos\theta$. Show that a Lagrangian for this system is

$$L(\theta, \dot{\theta}) = m(1 + \cos\theta)\dot{\theta}^2.$$

(iii) Find the general solution of the equation of motion and explain why the solutions are only valid for a finite time interval (excluding the trivial solutions $\theta$ =constant).

Hints: Obtain $\theta(t)$ by solving the first order ODE $H$ =constant and use the trigonometric identity $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$.

(iv) The Lagrangian

$$L(\theta, \dot{\theta}) = m(1 + \cos\theta)\dot{\theta}^2 + \frac{m\Omega^2}{2}(1 + \cos\theta)^2$$

describes a bead moving on a rotating wire where the constant $\Omega$ is the angular velocity of the rotating wire. Solve the equation of motion for the initial conditions $\theta(t = 0) = 0$ and $\dot{\theta}(t = 0) = \Omega$.

Hints: Fix the value of $H$ using the initial conditions and use the integral

$$\int \frac{du}{\cos u} = \ln\left[\tan\left(\frac{u}{2} + \frac{\pi}{4}\right)\right] + c.$$

[Total 30 marks]

**3.** (i) Use residues to compute the contour integrals

$$(a) \quad \oint_C \exp(1/z)(1+z+z^2)\ dz \qquad (b) \quad \oint_C \frac{\exp(1/z)}{z(1-qz)}\ dz,$$

where $C$ is the anti-clockwise oriented unit circle and $q$ is a complex constant ($|q| \neq 1$).

Hint: in part (b) consider the cases $|q| > 1$ and $|q| < 1$ separately.

(ii) The Bessel function $J_0$ is the entire function defined by

$$J_0(w) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{w}{2}\right)^{2m}, \tag{1}$$

where $w \in \mathbb{C}$.

(a) Show that

$$J_0(w) = \frac{1}{2\pi} \int_0^{2\pi} e^{iw\cos\theta} d\theta \quad (w \in \mathbb{C}). \tag{2}$$

Hint: Rewrite the integral in (2) as a contour integral over the unit circle (treating $w$ as a constant). Evaluate the contour integral using residues to obtain the series in (1).

(b) Compute the Fourier transform of $J_0(x)$.

Hint: Compute the Fourier transform of $e^{ix\cos\theta}$ (treating $\theta$ as a constant).

[Total 30 marks]

**4.** (i) Suppose

$$f(z) = \frac{1}{g(z)}$$

where $g$ is an analytic function.

Show that if $f$ has a simple pole at $z = w$ then Res $(f, w) = 1/g'(w)$.

(ii) Locate the poles and compute the associated residues for the complex function

$$f(z) = \frac{e^{-ikz}}{\cosh z}$$

where $k$ is a constant.

(iii) Show that the Fourier transform of

$$p(x) = \frac{1}{\cosh x} \quad \text{is} \quad \hat{p}(k) = \frac{1}{2\cosh(\frac{1}{2}\pi k)}.$$

Hint: Integrate $f(z)$ from part (ii) over the rectangular contour with vertices at $z = \pm L$ and $z = \pm L + i\pi$.

(iv) Express $p''(x)$ as a Fourier integral and use the result to compute the definite integral

$$\int_{-\infty}^{\infty} \frac{u^2}{\cosh u} du.$$

[Total 30 marks]

**5.** (i) Consider the Lagrangian

$$L = \frac{m}{2}\dot{x}_i\dot{x}_i + q\dot{x}_i A_i(\mathbf{r}),$$

where $m$ and $q$ are constants and $A_i$ is an arbitrary function of $x_1$, $x_2$ and $x_3$.

(a) Show that the Euler-Lagrange equations can be written in the form

$$m\ddot{x}_i = qF_{ij}\dot{x}_j,$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$. Use the equation of motion to show that

$$T = \frac{m}{2}\dot{x}_i\dot{x}_i$$

is a constant.

(b) Consider a constant $F_{ij}$ of the form

$$F_{ij} = i\mu(U_i U_j^* - U_j U_i^*),$$

where $U_i$ is a constant *complex* vector with the properties

$$U_i U_i = U_i^* U_i^* = 0, \quad U_i U_i^* = 1.$$

Here $\mu$ is a real constant and the $i$ before the $\mu$ is the imaginary unit. Consider a solution of the form

$$\dot{x}_i(t) = \mathrm{Re}(f(t)U_i),$$

where $f$ is a complex function of time. Find the general form of $f(t)$ and determine the frequency of oscillation.

Hint: For what $f(t)$ is the complex velocity $\dot{x}_i(t) = f(t)U_i$ a solution of the equation of motion?

(c) Verify that $T$ is constant for your solution.

(ii) The iterative map

$$x_{n+1} = \frac{x_n^2}{1+x_n},$$

is the Newton-Raphson process for finding the roots of a function $f$.

(a) Find $f(x)$. Is it unique?

(b) Does $x_n$ converge to a root of $f$ as $n \to \infty$ if $x_0 = 10$? For what range of $x_0$ values does $x_n$ not converge to a root?

[Total 30 marks]