Mathematical Methods (Spring term 2019)

Aims

The aim of the module is to study mathematical methods useful in Physics.

Examinable Content

The module is split into five parts:

1. CALCULUS OF VARIATIONS

Shortest curve joining two points, Euler-Lagrange equation as a stationarity condition on

$$S = \int_a^b L(y(x), y'(x), x)\ dx,$$

with $y(a)$ and $y(b)$ fixed. If L does not depend explicitly on x then

$$H = y' \frac{\partial L}{\partial y'} - L$$

is constant.

Hamilton's Principle (Lagrangian mechanics), $L = T - V$ for conservative forces, generalised coordinates and momenta, cyclic coordinates, Noether's theorem, Lagrangian for a charged particle using scalar and vector potentials.

Hanging rope problem, Lagrange multipliers, isoperimetric problems.

2. COMPLEX VARIABLES

Complex differentiation, analytic functions, Cauchy-Riemann equations, entire functions, harmonic property of analytic functions.

Complex integration, Fundamental Theorem of Calculus, Cauchy's theorem (anti derivative and standard form), Deformation theorem, Cauchy's integral formula and applications (proof of Liouville's theorem and proof that analytic functions are infinitely differentiable), isolated singularities and Laurent series, poles and essential singularities, meromorphic functions, Laurent's theorem.

Residue theorem and application to computing real integrals, Principal value of real and complex integrals, Residue theorem with simple poles on the contour.

3. FOURIER TRANSFORMS

Review of Fourier transforms and Fourier integrals. Properties of the Fourier transform (including Parseval formula and Convolution Theorem). Computation of Fourier transforms using contour integration.

Heaviside and sign function, delta function as a derivative, delta function as a limit of smooth functions, definition through the integral formula

$$\int_{-\infty}^{\infty} f(x)\ \delta(x-a)\ dx = f(a).$$

Properties of the delta function, Fourier transform (and Fourier integral representation) of the delta function.

Application of Fourier transforms to solving linear ODEs and PDEs (eg. driven oscillator ODEs and Laplace's equation in an infinite strip and half-plane).

4. TENSORS

Definition of vectors via their transformation properties, cartesian tensors, tensor algebra, contraction of tensor indices.

Vectors and pseudo-vectors (or polar and axial vectors), cartesian tensors, Levi-Civita symbol, cross product, grad, div, curl and Laplacian. Physical examples of cartesian tensors.

5 NUMERICAL METHODS

Numerical integration (trapezium rule and Simpson's rule), Newton-Raphson method, Runge-Kutta algorithm.

References

Calculus of Variations; see the chapter in Boas. Complex Variables; Boas and 'Complex Analysis' by H. A. Priestley. Tensors; Chapter 31 of 'The Feynman Lectures' volume 2.

Examination

2 hour paper with 5 questions (answer question 1 and *two* other questions. Question 1 comprises 8 short answer questions covering the whole module and is worth 40%. Questions 2 to 5 are worth 30% each.

Mathematical Proofs

The examination will predominantly be a test of technical skill rather than the ability to reproduce proofs. However, students are expected to be able to give *short* proofs (of results given in the lectures and problem sheets or simple results not seen previously). Students will not be asked for any long proofs, e.g. Taylor's theorem.