

Problem Sheet 7

1. *Lorentz Force*

Show that the Lagrangian

$$L = \frac{m}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t) - q\phi(\mathbf{r}, t),$$

leads to the Lorentz force law for a (non-relativistic) charged particle

$$m\ddot{\mathbf{r}} = q\mathbf{E}(\mathbf{r}, t) + q\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}, t).$$

The vector and scalar potentials, $\mathbf{A}(\mathbf{r}, t)$ and $\phi(\mathbf{r}, t)$, are related to the electric and magnetic fields, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, through

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

Hint: Check that the Euler-Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$$

yields the x -component of the Lorentz force law. Here ϕ , A_x , A_y and A_z are arbitrary functions of x , y , z and t .

2. A bead of mass m moves without friction on a helical wire. The helix can be parametrized as follows

$$x = R \cos u, \quad y = R \sin u, \quad z = \alpha u,$$

where u is a real parameter. Here α and R are constants. The acceleration due to gravity is the constant g (the gravitational force is pointing in the negative z direction). Obtain a Lagrangian $L(z, z')$ for the bead and solve the equation of motion.

3. A Version of Noether's Theorem

i) Suppose the Lagrangian $L(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t)$ is invariant¹ under the *time-independent* transformation

$$q'_i = q_i + \epsilon Q_i(q_1, q_2, \dots, q_n) \quad i = 1, \dots, n$$

where ϵ is 'small'. Show that

$$\sum_{i=1}^n p_i Q_i(q_1, q_2, \dots, q_n)$$

is a constant of the motion.

ii) A particle moving in two dimensions is described by the Lagrangian $L(x, y, \dot{x}, \dot{y}, t)$. The Lagrangian is invariant under the transformation

$$x' = x + \epsilon y, \quad y' = y - \epsilon x.$$

What do you conclude? Give an example of a Lagrangian with this property.

4. Rotating Pendulum Problem

A simple pendulum is mounted on a rotating turntable with constant angular velocity Ω . The pivot is on the axis of rotation.

i) Show that the kinetic energy of the pendulum bob is

$$T = \frac{ml^2}{2} (\dot{\theta}^2 + \Omega^2 \sin^2 \theta).$$

Here m is the mass of the bob, l is the length of the rod (assumed to be massless) and θ is the angle between the rod and the vertical. The potential energy is the same as for a non-rotating pendulum, ie. $V = -mgl \cos \theta$.

Hint: what is the kinetic energy of a particle in spherical polar coordinates?

ii) Obtain the equation of motion.

iii) Find all solutions of the form $\theta = \text{constant}$.

iv) Find a conserved quantity and show that $T + V$ is not a constant of the motion (unless θ is constant).

v) Determine the frequency of small oscillations about the constant θ solutions from part iii).

¹Here 'invariant' means $L(q'_1, q'_2, \dots, q'_n, \dot{q}'_1, \dot{q}'_2, \dots, \dot{q}'_n, t) = L(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t)$ up to terms of order ϵ^2 .