Mathematical Methods **Spring Term 2019**

Problem Sheet 4

1. Express $e^{-a|x|}$ as a Fourier integral ($a > 0$).

2. The function f is defined by

$$f(x) = \begin{cases} \cos x, & -\frac{1}{2}\pi < x < \frac{1}{2}\pi \\ 0, & \text{otherwise} \end{cases}$$

   Compute $\hat{f}(k)$. Sketch the graph of $\hat{f}(k)$.

3. Prove the following properties of the Fourier transform

   i) The Fourier transform of an even function is even.

   ii) The Fourier transform of a real odd function is imaginary.

   iii) $\widehat{f'}(k) = ik\hat{f}(k)$.

   iv) Acting with the Fourier transform four times reproduces the original function apart from an overall constant.

4. Use the standard integral

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right), \qquad (a > 0, \quad b \in \mathbb{C})$$

   to compute the Fourier transform of $f(x) = e^{-ax^2}$ and $g(x) = xe^{-ax^2}$ (here a is a positive constant).

5. Use residues to compute the Fourier transform of

$$f(x) = \frac{1}{1+x^4}.$$

   Hint: compute the cases $k > 0$ and $k < 0$ separately.

6. Write

$$\frac{\sin x}{x}$$

   as a Fourier integral.

7. Compute

   i)
$$\int_{-\infty}^{\infty} x^2\ \delta(x-3)\ dx$$
   ii)
$$\int_{-\infty}^{\infty} \delta(x^2+x)\ dx$$
   iii)
$$\int_{0}^{2} e^x\ \delta'(x-1)\ dx$$
   iv)
$$\int_{0}^{\infty} e^{-ax}\delta(\cos x)\ dx$$
   v)
$$\int_{0}^{\infty} \delta(e^{ax}\cos x)\ dx.$$

   In parts iv) and v) a is a constant.

8. Compute the derivatives

   i)
$$\frac{d}{dx}\left(\theta(x)\right)^3$$
   ii)
$$\frac{d}{dx}e^{a\theta(x)},$$

   where a is a constant.

9. i) Compute the Fourier transform of $\delta'(x-a)$ (a constant).

   ii) Express x^2 as a Fourier integral.

   iii) Write $\sin^2 x$ as a Fourier integral.

Fourier Transform Conventions

$$\text{Fourier transform} \quad \hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\ e^{-ikx}\ dx$$
$$\text{Fourier integral} \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(k)\ e^{ikx}\ dk.$$