Mathematical Methods **Spring Term 2015**

Answer to Problem Sheet 9

1. See appended printouts generated using Wolfram alpha.

2. i)

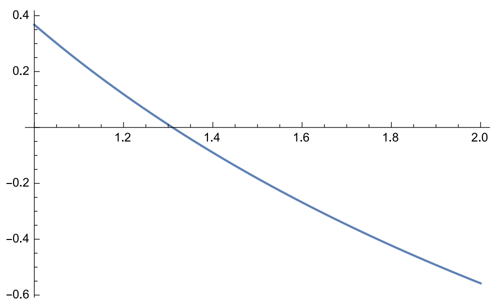
$$I \approx \frac{\pi}{12}\left(1 + 4\frac{\sin\frac{1}{4}\pi}{\frac{1}{4}\pi} + 2\frac{\sin\frac{1}{2}\pi}{\frac{1}{2}\pi} + 4\frac{\sin\frac{3}{4}\pi}{\frac{3}{4}\pi} + 0\right)$$

$$= \frac{\pi}{12}\left(1 + \frac{8\sqrt{2}}{\pi} + \frac{4}{\pi} + \frac{8\sqrt{2}}{3\pi}\right) = \frac{\pi}{12} + \frac{1}{3} + \frac{8\sqrt{2}}{9} \approx 1.852.$$

   ii) See appended printouts generated using Wolfram alpha.

3. Show graphically that the equation $e^{-x} = \ln x$ has a solution with $1 < x < 2$. Hence solve the equation to 4 decimal places by the Newton-Raphson method.

   Let $f(x) = e^{-x} - \ln x$. Graph of function:



   From the graph this has a root close to 1.3.

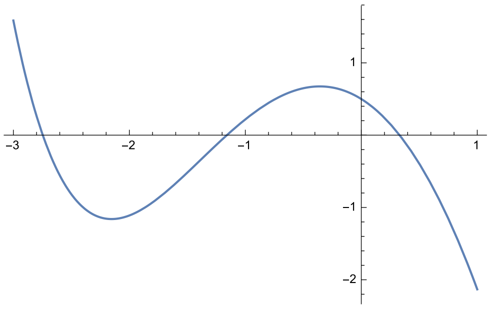
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{-x_n} - \ln x_n}{-e^{-x_n} + 1/x_n} =$$

   Set $x_0 = 1.3$. $x_1 = 1.3098$. It is not necessary to proceed to x_2 unless higher accuracy is required.

4. Show graphically that the equation $e^{-x} = 2x^2 + \frac{1}{2}$ has three solutions which lie between -3 and 1. Solve the equation to 2 decimal places by the

   Let $f(x) = e^{-x} - 2x^2 - \frac{1}{2}$: Graph of function:

   Graph has roots near $x = -2.8, -1.1, 0.3$



Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{-x_n} - 2x_n^2 - \frac{1}{2}}{-e^{-x_n} - 4x_n}.$$

Applying one iteration to each of the three 'guesses' in turn gives -2.746, -1.160, 0.331.

5. Use the simple Runge-Kutta process

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2), \qquad k_1 = f(x_n, y_n), \qquad k_2 = f(x_{n+1}, y_n + hk_1)$$

to find $y(1)$ for the equation

$$\frac{dy}{dx} = f(x, y) = y$$

subject to $y(0) = 1$, choosing $h = 0.5$. Compare your answer with the exact solution. Repeat the calculation with $h = 0.25$. Work to three decimal places.

$k_1 = y_n$, $k_2 = y_n + hk_1 = (1 + h)y_n$.

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2) = y_n(1 + h + \frac{1}{2}h^2).$$

$h = 0.5$. $y_{n+1} = \frac{13}{8}y_n$. $y_2 = \frac{169}{64}y_0 = 2.640$.

$h = 0.25$, $y_{n+1} = \frac{41}{32}y_n$, $y_4 = (\frac{41}{32})^4 y_0 = 2.695$

Exact result $y(x) = e^x$. $y(0) = e = 2.718...$

6. Use Runga-Kutta to find $y(0.2)$ for the equation

$$\frac{dy}{dx} = f(x, y) = 10x^2 + y$$

subject to $y(0) = 2$, choosing the interval $h = 0.1$. Work to 4 decimal places. Compare your answer with the exact solution.

$k_1 = 10x_n^2 + y_n = 10h^2n^2 + y_n$, $k_2 = 10h^2x_{n+1}^2 + y_n + hk_1 = 10h^2(n+1)^2 + y_n + hk_1$

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$$

$y_0 = 2$

$y_1 = 2 + \frac{1}{20}(k_1 + k_2)$ with $k_1 = 2$, $k_2 = \frac{1}{10} + 2 + \frac{1}{10} \cdot 2 = 2.3$ so that $y_1 = 2 + \frac{1}{20}(2 + 2.3) = 2.215$

$y_2 = 2.215 + \frac{1}{20}(k_1 + k_2)$ with $k_1 = \frac{1}{10} + y_1 = 2.315$, $k_2 = 0.4 + y_1 + \frac{1}{10}k_1 = 0.4 + 2.215 + 0.2315 = 2.8465$ so that $y_2 = 2.215 + \frac{1}{20}(2.315 + 2.8465) = 2.473$.

Exact solution $y(x) = 22e^x - 10(x^2 + 2x + 2)$ so that $y(0.2) = 2.471$.

7. i) Solve the following set of equations by Gaussian elimination

$$\begin{aligned} x + 2y - z &= 4 \\ 2x + y + z &= 5 \\ 2x - y + 2z &= 2. \end{aligned}$$

Answer $x = 1$, $y = 2$, $z = 1$.

ii) Use the Gauss-Jordan technique to find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 2 & -1 & 2 \end{pmatrix}.$$

Answer

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & -3 & 3 \\ -2 & 4 & -3 \\ -4 & 5 & -3 \end{pmatrix}.$$

8 step trapezoidal rule exp(-x^2) on [0,1]

Input interpretation:

integrate	$\exp(-x^2)$	using trapezoidal rule	with 8 intervals; from $x = 0$ to 1

Result:

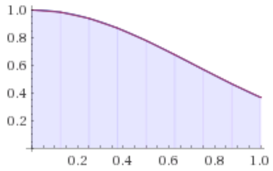
0.745866

Symbolic form of trapezoidal rule:

Show details

$$\int_0^1 e^{-x^2}\,dx \approx \frac{1}{16}\sum_{n=0}^{7}\left(e^{-\frac{n^2}{64}} + e^{-\frac{1}{64}(1+n)^2}\right)$$

Plot:



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Exact result:

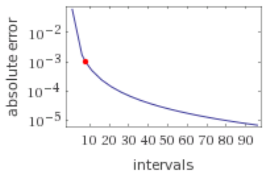
More digits

$$\frac{1}{2}\sqrt{\pi}\,\operatorname{erf}(1) \approx 0.7468241328124270$$

$\operatorname{erf}(x)$ is the error function

Error vs. intervals:

Relative error



Method comparisons:

method	result	absolute error	relative error
left endpoint	0.785373	0.038549	0.0516173
right endpoint	0.706358	0.0404661	0.0541842
midpoint	0.747304	0.000479446	0.00064198
trapezoidal rule	0.745866	0.000958518	0.00128346
Simpson's rule	0.746824	1.24623×10^{-7}	1.66871×10^{-7}
Boole's rule	0.746824	5.97522×10^{-12}	8.00084×10^{-12}

Mathematica input:

8 step trapezoidal rule 1/(1+x^4) on [0,1]

Examples Random

Input interpretation:

integrate	$\frac{1}{1+x^4}$	using trapezoidal rule	with 8 intervals from $x = 0$ to 1

Result:

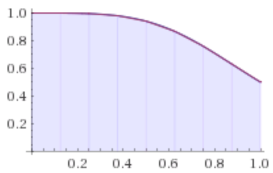
0.865669

Symbolic form of trapezoidal rule:

Show details

$$\int_0^1 \frac{1}{1+x^4}\,dx \approx \frac{1}{16}\sum_{n=0}^{7}\left(\frac{1}{1+\frac{n^4}{4096}} + \frac{1}{1+\frac{(1+n)^4}{4096}}\right)$$

Plot:



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Exact result:

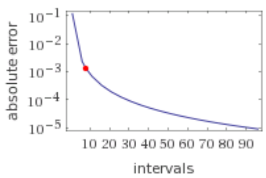
More digits

$$\frac{\pi + 2\coth^{-1}(\sqrt{2})}{4\sqrt{2}} \approx 0.8669729873399110$$

$\coth^{-1}(x)$ is the inverse hyperbolic cotangent function

Error vs. intervals:

Relative error



Method comparisons:

method	result	absolute error	relative error
left endpoint	0.896919	0.0299459	0.0345407
right endpoint	0.834419	0.0325541	0.0375492
midpoint	0.867626	0.000652821	0.000752989
trapezoidal rule	0.865669	0.00130412	0.00150422
Simpson's rule	0.866973	5.08364×10^{-7}	5.86367×10^{-7}
Boole's rule	0.866973	1.64525×10^{-11}	1.8977×10^{-11}

Mathematica input:

x

2 step simpson x^2/(1+x^2) on [0,1]

Examples Random

Input interpretation:

integrate	$\frac{x^2}{1+x^2}$	using Simpson's rule	with 2 intervals from $x = 0$ to 1

Result:

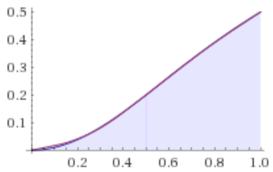
0.214608

Symbolic form of Simpson's rule:

Show details

$$\int_0^1 \frac{x^2}{1+x^2}\,dx \approx \frac{1}{12}\left(0 + 2\sum_{n=1}^{2-1}\frac{n^2}{4+n^2} + 4\sum_{n=1}^{2}\frac{(1-2n)^2}{17-4n+4n^2} + \frac{1}{2}\right)$$

Plot:



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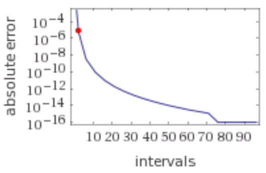
Exact result:

More digits

$$1 - \frac{\pi}{4} \approx 0.2146018366025517$$

Error vs. intervals:

Relative error



(in machine arithmetic)

Method comparisons:

method	result	absolute error	relative error
left endpoint	0.1	0.114602	0.534021
right endpoint	0.35	0.135398	0.630927
midpoint	0.209412	0.00519007	0.0241847
trapezoidal rule	0.225	0.0103982	0.0484533
Simpson's rule	0.214608	6.00653×10^{-6}	0.0000279892
Boole's rule	0.214601	3.60134×10^{-7}	1.67815×10^{-6}

Mathematica input: