## Answers to Problem Sheet 7

1.
$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t) - q\phi(\mathbf{r},t).$$

Expanding out the dot products the Lagrangian can be written

$$L = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) + q\left(A_x\dot{x}+A_y\dot{y}+A_z\dot{z}-\phi\right).$$

Therefore
$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x.$$

To obtain the Lorentz force law one must regard $\phi$, $A_x$, $A_y$ and $A_z$ as arbitrary functions of $x$, $y$, $z$ and $t$. Accordingly,

$$\frac{\partial L}{\partial x} = q\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z} - \frac{\partial \phi}{\partial x}\right)$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} + q\frac{dA_x}{dt} = m\ddot{x} + q\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} + \frac{\partial A_x}{\partial t}\right).$$

This gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} + q\left[\frac{\partial \phi}{\partial x} + \frac{\partial A_x}{\partial t} + \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)\dot{y} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_x}{\partial z}\right)\dot{z}\right]$$
$$= m\ddot{x} + q[-E_x - B_z\dot{y} + B_y\dot{z}] = 0,$$

which is the $x$-component of the Lorentz force law. Similarly, the other two Euler-Lagrange equations yield the $y$ and $z$ components of the Lorentz force law.

2.
$$x = R\cos u, \quad y = R\sin u \quad z = \alpha u,$$

where $u$ is a real parameter.

$$T = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) = \frac{m}{2}(R^2\sin^2 u\,\dot{u}^2 + R^2\cos^2 u\,\dot{u}^2 + \dot{z}^2) = \frac{m}{2}(R^2\dot{u}^2+\dot{z}^2) =$$
$$= \frac{m}{2}\left(\frac{R^2}{\alpha^2}+1\right)\dot{z}^2.$$

The potential energy is $V = mgz$. The Lagrangian is

$$L = T - V = \frac{m}{2}\left(\frac{R^2}{\alpha^2}+1\right)\dot{z}^2 - mgz.$$

The equation of motion can be written as

$$\ddot{z} = -\tilde{g}, \qquad \tilde{g} = \frac{\alpha^2 g}{R^2+\alpha^2}.$$

This is the same as for vertical motion under constant gravity but with a reduced acceleration $\tilde{g}$.

3. i)

$$L(q_1', q_2', ..., q_n', \dot{q}_1', ..., \dot{q}_n', t) = L(q_1, q_2, ..., q_n, \dot{q}_1, ..., \dot{q}_n, t)$$
$$+ \sum_{i=1}^{n} \left( \frac{\partial L}{\partial q_i} \epsilon Q_i(q_1, q_2, ..., q_n) + \frac{\partial L}{\partial \dot{q}_i} \epsilon \frac{d}{dt} Q_i(q_1, q_2, ..., q_n) \right)$$

Now $p_i = \partial L/\partial \dot{q}_i$ and the Euler-Lagrange equations can be written as $\dot{p}_i = \partial L/\partial q_i$. Accordingly

$$L(q_1', q_2', ..., q_n', \dot{q}_1', ..., \dot{q}_n', t) - L(q_1, q_2, ..., q_n, \dot{q}_1, ..., \dot{q}_n, t) = \epsilon \frac{d}{dt} \sum_{i=1}^{n} p_i Q_i(q_1, q_2, ..., q_n).$$

If the Lagrangian is invariant under the transformation

$$\sum_{i=1}^{n} p_i Q_i(q_1, q_2, ..., q_n)$$

is a constant of the motion.

ii) Apply Noether's theorem with $q_1 = x$, $q_2 = y$, $Q_1 = y = q_2$, $Q_2 = -x = -q_1$ then

$$p_1 Q_1 + p_2 Q_2 = y p_x - x p_y$$

is a constant of the motion. This is conservation of angular momentum $L_z = x p_y - y p_z$.

The transformation is an infinitesimal rotation about the origin. Any Lagrangian that is rotationally invariant has this symmetry (eg. the Kepler problem) and associated conservation law.

4. i) The kinetic energy of a particle of mass $m$ in spherical polar coordinates is

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right).$$

Spherical polar coordinates are conventionally defined with $\theta$ as the angle between $\hat{\mathbf{r}}$ and the positive $z$-axis so that $z = r\cos\theta$. Now let $\theta$ be measured with respect to the negative $z$-axis so that $z = -r\cos\theta$ and as usual $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$. With these 'modified' spherical polar coordinates the kinetic energy formula is unchanged.

Now setting $r = l$ and $\dot{\phi} = \Omega$ gives the stated formula

$$T = \frac{ml^2}{2}\left(\dot{\theta}^2 + \Omega^2\sin^2\theta\right).$$

ii) The potential energy is the same as for a non-rotating pendulum, ie. $V = -mgl\cos\theta$. Accordingly,

$$L = T - V = \frac{ml^2}{2}\left(\dot{\theta}^2 + \Omega^2\sin^2\theta\right) + mgl\cos\theta.$$

The Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

yields

$$ml^2\ddot{\theta} - ml^2\Omega^2\sin\theta\cos\theta + mgl\sin\theta = 0,$$

or

$$\ddot{\theta} = \sin\theta\left(-\frac{g}{l} + \Omega^2\cos\theta\right).$$

iii) Solutions of the form $\theta$ =constant require $\sin\theta = 0$ (yielding the standard equilibria $\theta = 0$ and $\theta = \pi$) or

$$\left(-\frac{g}{l} + \Omega^2\cos\theta\right) = 0$$

which has the solutions

$$\theta = \pm\cos^{-1}\frac{g}{\Omega^2 l}$$

provided $\Omega^2 > g/l$.

iv) As $L$ does not depend explicitly on $t$

$$H = \dot{\theta}\frac{\partial L}{\partial\dot{\theta}} - L = \frac{ml^2}{2}\dot{\theta}^2 - \frac{ml^2}{2}\Omega^2\sin^2\theta - mgl\cos\theta$$

is constant. Note that this is *not* the total energy

$$T + V = \frac{ml^2}{2}\dot{\theta}^2 + \frac{ml^2}{2}\Omega^2 \sin^2\theta - mgl\cos\theta.$$

The total energy is not constant since

$$\frac{d}{dt}(T+V) = \frac{d}{dt}(H + ml^2\Omega^2 \sin^2\theta) = 2ml^2\Omega^2 \sin\theta\cos\theta\,\dot{\theta}.$$

v) To find the frequency of small oscillations look at the equations of motion for $\theta$ near $\theta = 0, \pi$ and $\theta_0 = \cos^{-1}(g/\Omega^2 l)$.

Near $\theta = 0$

$$\ddot{\theta} = \theta\left(-\frac{g}{l} + \Omega^2\right),$$

a SHO with angular frequency $\omega = \sqrt{g/l\ - \Omega^2}$. If $\Omega^2 > g/l, \quad \theta = 0$ is an unstable equilibrium point.

For $\theta$ near $\theta_0$

$$\frac{d^2}{dt^2}(\theta - \theta_0) = \sin[\theta_0 + (\theta - \theta_0)]\left(-\frac{g}{l} + \Omega^2\cos[\theta_0 + (\theta - \theta_0)]\right) \approx -\Omega^2 \sin^2\theta_0\,(\theta - \theta_0),$$

using the Taylor expansion $\cos[\theta_0 + (\theta - \theta_0)] = \cos\theta_0 - \sin\theta_0(\theta - \theta_0) + \ldots$. The approximate equation of motion describes a SHO with

$$\omega^2 = \Omega^2\sin^2\theta_0 = \Omega^2(1 - \cos^2\theta_0) = \Omega^2\left(1 - \frac{g^2}{l^2\Omega^4}\right)$$

or

$$\omega = \Omega\sqrt{1 - \frac{g^2}{l^2\Omega^4}},$$

which is always less than the frequency of the turntable.

The equilibrium at $\theta = \pi$ is unstable for any $\Omega$. Why?