Answers to Problem Sheet 5

1. i) $f(x) = e^{ax}\theta(x)$, $g(x) = e^{bx}\theta(x)$.

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_0^{\infty} e^{at}g(x-t)dt.$$

This is zero if $x < 0$ since then $g(x-t)$ is zero for all positive t. If $x > 0$

$$(f\star g)(x) = \int_0^x e^{at}e^{b(t-x)}dt = e^{bx}\int_0^x e^{t(a-b)}dt = e^{bx}\,\frac{e^{t(a-b)}}{a-b}\bigg|_{t=0}^{t=x} = \frac{e^{ax}-e^{bx}}{a-b}.$$

Accordingly,

$$(f \star g)(x) = \frac{e^{ax}-e^{bx}}{a-b}\;\theta(x).$$

What happens if $a = b$?

ii) $f(x) = 1/(x^2+a^2)$ and $g(x) = 1/(x^2+b^2)$. It is messy to compute $(f\star g)(x)$ directly (try it!). Following the hint use $(\widehat{f \star g})(k) = 2\pi\hat{f}(k)\hat{g}(k)$. We require $\hat{f}(k)$ ($\hat{g}(k)$ is the same with a replaced by b). Assume a and b are positive. Now

$$\hat{f}(k) = \frac{e^{-a|k|}}{2a}.$$

This can be obtained using the Fourier integral from Q1 on Problem Sheet 6 or by contour integration. Therefore

$$(\widehat{f \star g})(k) = 2\pi\cdot\frac{e^{-a|k|}}{2a}\cdot\frac{e^{-b|k|}}{2b} = \frac{\pi(a+b)}{ab}\cdot\frac{e^{-(a+b)|k|}}{2(a+b)},$$

which is $\pi(a+b)/(ab)$ multiplied by the Fourier transform of $1/[x^2+(a+b)^2]$. Therefore

$$(f \star g)(x) = \frac{\pi(a+b)}{ab}\cdot\frac{1}{x^2+(a+b)^2}.$$

2. i) Multiply by $f(x)$ and integrate from $x = -\infty$ to $x = \infty$. This gives (on switching the order of the summation and integration)

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{inx}dx = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\delta(x-2\pi m)dx,$$

or

$$\sum_{n=-\infty}^{\infty} \hat{f}(-n) = \sum_{m=-\infty}^{\infty} f(2\pi m).$$

As the sum is over all integers one can replace $\hat{f}(-n)$ with $\hat{f}(n)$.

ii) Take $f(x) = e^{-a|x|}$ with $a > 0$ ($f(x) = [a^2 + (x/2\pi)^2]^{-1}$ also works). From Q1 on Problem Sheet 6

$$\hat{f}(k) = \frac{a}{\pi(k^2 + a^2)}.$$

Using Poisson's summation formula

$$\frac{a}{\pi} \sum_{p=-\infty}^{\infty} \frac{1}{p^2 + a^2} = \sum_{m=-\infty}^{\infty} f(2\pi m) = 1 + 2\sum_{m=1}^{\infty} e^{-2\pi m a}$$

$$= 1 + \frac{2e^{-2\pi a}}{1 - e^{-2\pi a}} = \frac{1 + e^{-2\pi a}}{1 - e^{-2\pi a}} = \frac{e^{\pi a} + e^{-\pi a}}{e^{\pi a} - e^{-\pi a}} = \coth \pi a$$

so that

$$\sum_{p=-\infty}^{\infty} \frac{1}{p^2 + a^2} = \frac{\pi \coth \pi a}{a}.$$

3. i) $f(x) = \delta'(x - a)$ (a constant).

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \delta'(x - a) dx$$

$$= \frac{1}{2\pi} \left[e^{-ikx} \delta(x - a) \Big|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} (-ik) e^{-ikx} \delta(x - a)\, dx \right] = \frac{ike^{-ika}}{2\pi}.$$

ii)

$$x^2 = -\int_{-\infty}^{\infty} \delta''(k) e^{ikx} dk.$$

iii)

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{e^{2ix}}{4} - \frac{e^{-2ix}}{4}$$

$$= \int_{-\infty}^{\infty} \left[\frac{\delta(k)}{2} - \frac{\delta(k-2)}{4} - \frac{\delta(k+2)}{4} \right] e^{ikx} dk.$$

4. *i)* $\ddot{x}(t) + 4x(t) = \sin t / t$. Write $\sin t/t$ as a Fourier integral (see problem sheet 6)

$$\frac{\sin t}{t} = \frac{1}{2} \int_{-1}^{1} e^{i\omega t} d\omega.$$

A particular solution is

$$x_{PI}(t) = \frac{1}{2}\int_{-1}^{1} \frac{e^{i\omega t}}{-\omega^2+4} d\omega.$$

ii) $\ddot{x}(t) + 2\dot{x}(t) + x(t) = \delta(t)$. Write $\delta(t)$ as a Fourier integral

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} d\omega$$

Therefore a particular solution is

$$x_{PI}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t} d\omega}{-\omega^2+2i\omega+1} = \frac{1}{2\pi}\oint_C f(z)\ dz$$

where $f(z) = e^{izt}/(-z+2iz+1) = -e^{izt}/(z-i)^2$ and C is a semi-circular contour with radius $R > 1$ (if $t > 0$ the semi-circle must be taken in the upper half-plane and if $t < 0$ in the lower half-plane).

This has a double pole at $z = i$. To compute the residue Taylor expand the exponential about $z = i$. $e^{izt} = e^{i[(z-i)+i]t} = e^{-t}[1 + it(z-i) + ...]$ so that $\text{Res}(f, i) = -ite^{-t}$.

If $t < 0$ the contour does not enclose the pole and so $x_{PI}(t) = 0$. If $t > 0$ the Residue Theorem gives $x_{PI}(t) = te^{-t}$. One can combine the two results $x_{PI}(t) = te^{-t}\theta(t)$. Check that this satisfies the ODE!!

5.

$$\phi(x,y) = \int_{-\infty}^{\infty} \hat{f}(k)\ \frac{e^{ikx}\sinh ky}{\sinh k}\ dk,$$

The Laplacian of ϕ is

$$\phi_{xx}+\phi_{yy} = \int_{-\infty}^{\infty} \hat{f}(k)\ (ik)^2\frac{e^{ikx}\sinh ky}{\sinh k}\ dk + \int_{-\infty}^{\infty} \hat{f}(k)\ \frac{e^{ikx}k^2\sinh ky}{\sinh k}\ dk = 0,$$

so that ϕ is harmonic. Since $\sinh ky = 0$ if $y = 0$, $\phi(x, y = 0) = 0$. At $y = 1$

$$\phi(x, y=1) = \int_{-\infty}^{\infty} \hat{f}(k)\ \frac{e^{ikx}\sinh k}{\sinh k}\ dk = \int_{-\infty}^{\infty} \hat{f}(k)\ e^{ikx}\ dk = f(x).$$

With the boundary conditions

$$\phi(x,0) = 0, \quad \phi(x,1) = e^{-\frac{1}{2}x^2}$$

$$\phi(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}k^2}\ \frac{e^{ikx}\sinh ky}{\sinh k}\ dk,$$

since $\hat{f}(k) = e^{-\frac{1}{2}k^2}/\sqrt{2\pi}$ (see problem sheet 6).

ii) $\phi(x, y)$ is harmonic in half-plane $-\infty < x < \infty$, $y \geq 0$ with the properties

$$\phi(x, y = 0) = e^{-|x|}, \quad \phi(x, y) \to 0 \text{ as } y \to \infty.$$

The (harmonic) function $\phi(x, y) = e^{ikx-|k|y}$ decays exponentially as $y \to \infty$. Consider a linear combination of these solutions

$$\phi(x, y) = \int_{-\infty}^{\infty} c(k) e^{ikx-|k|y} dk$$

The $y = 0$ boundary condition gives $c(k) = \hat{f}(k)$ where $f(x) = e^{-|x|}$. From problem sheet 6, $\pi \hat{f}(k) = 1/(1 + k^2)$ giving

$$\phi(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx-|k|y}}{1 + k^2} dk.$$

6. Writing $\phi(x, t)$ as a Fourier integral

$$\phi(x, t) = \int_{-\infty}^{\infty} A(k, t) e^{ikx} dk,$$

where $A(k, t)$ is the Fourier transform of $\phi(x, t)$ with respect to x only (t is not Fourier-transformed).

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \int_{-\infty}^{\infty} \left(-k^2 A(k, t) - \frac{1}{c^2} A_{tt}(k, t) \right) e^{ikx} dk$$

The wave equation yields

$$A_{tt}(k, t) = -c^2 k^2 A(k, t)$$

which has the general solution

$$A(k, t) = p(k) e^{ickt} + q(k) e^{-ickt},$$

where the 'constants of integration' p and q are arbitrary functions of k. Therefore

$$\phi(x, t) = \int_{-\infty}^{\infty} \left(p(k) e^{ickt} + q(k) e^{-ickt} \right) e^{ikx} dk$$

$$= \int_{-\infty}^{\infty} p(k) e^{ik(x+ct)} dk + \int_{-\infty}^{\infty} q(k) e^{ik(x-ct)} dk = f(x + ct) + g(x - ct),$$

where $p(k) = \hat{f}(k)$ and $q(k) = \hat{g}(k)$.

Fourier Transform Conventions

$$\text{Fourier transform} \quad \hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\ e^{-ikx}\ dx$$

$$\text{Fourier integral} \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(k)\ e^{ikx}\ dk.$$