Mathematical Methods **Spring Term 2019**

Answers to Problem Sheet 4

1. $f(x) = e^{-a|x|}$. Fourier transform

$$\hat{f}(k = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\ dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a|x|}e^{-ikx}\ dx$$
$$= \frac{1}{2\pi}\left[\int_0^{\infty} e^{-ax}e^{-ikx}\ dx + \int_{-\infty}^{0} e^{+ax}e^{-ikx}\ dx\right].$$

Combining the exponentials yields

$$\hat{f}(k) = \frac{1}{2\pi}\left[\int_0^{\infty} e^{x(-a-ik)}\ dx + \int_{-\infty}^{0} e^{x(a-ik)}\ dx\right].$$
$$= \frac{1}{2\pi}\left[\left.\frac{e^{x(-a-ik)}}{-a-ik}\right|_{x=0}^{x=\infty} + \left.\frac{e^{x(a-ik)}}{a-ik}\right|_{x=-\infty}^{x=0}\right]$$
$$= \frac{1}{2\pi}\left[\frac{1}{a+ik} + \frac{1}{a-ik}\right] = \frac{a}{\pi(a^2+k^2)}.$$

Fourier integral

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\ dk = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{a^2+k^2}\ dk$$

2.

$$f(x) = \begin{cases} \cos x, & -\frac{1}{2}\pi < x < \frac{1}{2}\pi \\ 0, & \text{otherwise} \end{cases}$$

Using

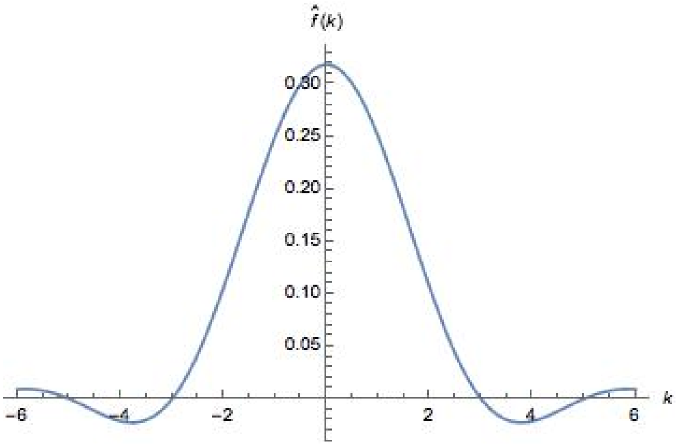
$$\cos x = \frac{e^{ix}+e^{-ix}}{2}.$$

$$2\pi\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos x\, e^{-ikx}dx = \frac{1}{2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left(e^{x(i-ik)} + e^{x(-1-ik)}\right)dx$$
$$= \frac{1}{2}\left.\frac{e^{x(i-ik)}}{i(1-k)}\right|_{x=-\frac{1}{2}\pi}^{x=\frac{1}{2}\pi} + \frac{1}{2}\left.\frac{e^{x(-i-ik)}}{i(-1-k)}\right|_{x=-\frac{1}{2}\pi}^{x=\frac{1}{2}\pi}$$

Using $e^{\frac{1}{2}i\pi} = i$ and $e^{-\frac{1}{2}i\pi} = -i$ this reduces to

$$\hat{f}(k) = \frac{e^{-\frac{1}{2}i\pi k} + e^{i\frac{1}{2}\pi k}}{4\pi}\left(\frac{1}{1-k} + \frac{1}{1+k}\right) = \frac{\cos\left(\frac{\pi k}{2}\right)}{\pi(1-k^2)}$$

Sketch: note that $\hat{f}(k)$ is well behaved at $k = \pm 1$. In fact, $\hat{f}(k = \pm 1) = \frac{1}{4}$. Why? $\hat{f}(k)$ has a (global) maximum at $k = 0$; $\hat{f}(k = 0) = 1/\pi$



3. i)

$$\hat{f}(-k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx.$$

Let $u = -x$ so that $dx = -du$

$$\hat{f}(-k) = \frac{1}{2\pi} \int_{\infty}^{-\infty} f(-u) e^{-iku} (-du) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(-u) e^{-iku} du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{-iku} du = \hat{f}(k)$$

if f is even.

ii) If f is odd $f(-u) = -f(u)$ so the calculation from part i) gives $\hat{f}(-k) = -\hat{f}(k)$ so that $\hat{f}$ is odd. To establish that $\hat{f}$ is imaginary for f odd and real consider $\hat{f}^{\,*}$:

$$\hat{f}^{\,*}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx = \hat{f}(-k) = -\hat{f}(k),$$

so that $\hat{f}(k)$ is purely imaginary.

iii)

$$\widehat{f'}\,(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f'(x) e^{-ikx} dx = \frac{1}{2\pi} \left[e^{-ikx} f(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -ike^{ikx} f(x) dx \right] = ik\hat{f}(k).$$

using integration by parts (the boundary term vanishes if $f(x) \to 0$ as $x \to \pm\infty$).

iv)

$$f(-x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{-ikx} dk = 2\pi \hat{\hat{f}}\,(x).$$

Let $g(x) = \hat{\hat{f}}\,(x)$

$$g(-x) = 2\pi\,\hat{\hat{g}}(x) = 2\pi\hat{\hat{\hat{\hat{f}}}}\,(x).$$

But $f(x) = 2\pi g(-x)$ so that

$$f(x) = (2\pi)^2\hat{\hat{\hat{\hat{f}}}}\,(x).$$

4. $f(x) = e^{-ax^2}$

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ax^2}e^{-ikx}\,dx$$

Using the integral formula from part i) with $b = -ik$

$$\hat{f}(k) = \frac{1}{\sqrt{4\pi a}}e^{-k^2/(4a)}.$$

$g(x) = xe^{-ax^2} = f'(x)/(-2a)$ so that $\hat{g}(k) = -ik\hat{f}(k)/(2a)$.

5.

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ikx}}{1+x^4}dx.$$

Consider $g(z) = e^{-ikz}/(1+z^4)$. If $k < 0$ take a semi-circular contour in the upper half-plane of radius R (as in the previous problem sheet). If $R > 1$ two poles are enclosed by the contour. The residues are

$$\text{Res}(g, e^{i\pi/4}) = -\frac{e^{i\pi/4}}{4}e^{-ike^{i\pi/4}} = -\frac{(1+i)}{4\sqrt{2}}e^{k(1-i)/\sqrt{2}}$$

$$\text{Res}(g, e^{3\pi i/4}) = \frac{e^{-i\pi/4}}{4}e^{-ike^{3\pi i/4}} = \frac{(1-i)}{4\sqrt{2}}e^{k(1+i)/\sqrt{2}}.$$

On taking the $R \to \infty$ limit one finds (for $k < 0$)

$$\hat{f}(k) = \frac{1}{2\pi}\cdot 2\pi i\left[-\frac{(1+i)}{4\sqrt{2}}e^{k(1-i)/\sqrt{2}} + \frac{(1-i)}{4\sqrt{2}}e^{k(1+i)/\sqrt{2}}\right]$$

$$= \frac{e^{k/\sqrt{2}}}{4\sqrt{2}}\left[(1-i)e^{-ik/\sqrt{2}} + (1+i)e^{ik/\sqrt{2}}\right].$$

To compute $\hat{f}(k)$ for $k > 0$ take the semi-circle in the lower half plane or simply exploit the result that the Fourier transform of an even function is even.

One can also eliminate i from the formula:

$$\hat{f}(k) = \frac{e^{-|k|/\sqrt{2}}}{2} \cos\left[\frac{|k|}{\sqrt{2}} - \frac{\pi}{4}\right].$$

6. $f(x) = \sin x/x$ with Fourier transform

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin x}{x} e^{-ikx} dx.$$

This integral resembles the Fourier integral for the 'square pulse' $g(x) = 1$ for $|x| < 1$ and $g(x) = 0$ otherwise:

$$g(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin k}{k} e^{ikx} dx.$$

Evidently $\hat{f}(k) = \frac{1}{2}g(-k)$ giving the Fourier integral

$$f(x) = \frac{1}{2}\int_{-1}^{1} e^{ikx} dk.$$

7. i)

$$\int_{-\infty}^{\infty} x^2\ \delta(x-3)\ dx = 9$$

ii) $h(x) = x^2 + 2 = x(x+1)$ with roots $x_1 = 0$ and $x_2 = -1$. $h'(x) = 2x+1$ so that $|h'(x_1)| = 1$ and $|h'(x_2)| = 1$. Therefore $\delta(x^2+x) = \delta(x) + \delta(x+1)$

$$\int_{-\infty}^{\infty} \delta(x^2+x)\ dx = 2$$

iii)

$$\int_0^2 e^x\ \delta'(x-1)\ dx = e^x\delta(x-1)|_0^2 - \int_0^2 e^x\delta(x-1)dx = -e$$

iv) $h(x) = \cos x$ has root at odd integer multiples of $\pi/2$ and the derivatives have absolute value 1 at the roots. Hence

$$\delta(\cos x) = \sum_{n \text{ odd}} \delta(x - \tfrac{1}{2}n\pi).$$

To compute the integral

$$\int_0^{\infty} e^{-ax}\delta(\cos x)\ dx$$

only positive n terms count due to the range of integration. Accordingly

$$\int_0^\infty e^{-ax}\delta(\cos x)\ dx = \sum_{n \text{ odd}} e^{-\frac{1}{2}n\pi a} = e^{-\frac{1}{2}\pi a}\sum_{p=0}^{\infty} e^{-p\pi a} = \frac{1}{2\sinh\frac{1}{2}\pi a},$$

assuming a is positive.

v)

$$\int_0^\infty \delta(e^{ax}\cos x)\ dx.$$

Here $h(x) = e^{ax}\cos x$ with roots (as in part iv)) $x_n = \frac{1}{2}\pi n$ where n is an odd integer. $h'(x) = ae^{ax}\cos x - e^{ax}\sin x$ giving $|h'(x_n)| = e^{\frac{1}{2}n\pi a}$. The integral yields the same infinite sum as in part iv).

8. i)

$$\frac{d}{dx}\left(\epsilon(x)\right)^3 = \frac{d}{dx}\epsilon(x) = 2\delta(x)$$

ii) $e^{a\theta(x)} = e^a$ if $x > 0$ and $e^{a\theta(x)} = 1$ if $x < 0$. Therefore

$$e^{a\theta(x)} = (e^a - 1)\theta(x) + 1.$$

Accordingly,

$$\frac{d}{dx}e^{a\theta(x)} = (e^a - 1)\delta(x).$$

9. i) $f(x) = \delta'(x-a)$ (a constant).

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\delta'(x-a)dx$$

$$= \frac{1}{2\pi}\left[e^{-ikx}\delta(x-a)\Big|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty}(-ik)e^{-ikx}\delta(x-a)\ dx\right] = \frac{ike^{-ika}}{2\pi}.$$

ii)

$$x^2 = -\int_{-\infty}^{\infty}\delta''(k)e^{ikx}dk.$$

iii)

$$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{e^{2ix}}{4} - \frac{e^{-2ix}}{4}$$

$$= \int_{-\infty}^{\infty}\left[\frac{\delta(k)}{2} - \frac{\delta(k-2)}{4} - \frac{\delta(k+2)}{4}\right]e^{ikx}dk.$$

Fourier Transform Conventions

$$\text{Fourier transform} \quad \hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\; e^{-ikx}\; dx$$

$$\text{Fourier integral} \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(k)\; e^{ikx}\; dk.$$