University College London

Department of Physics and Astronomy

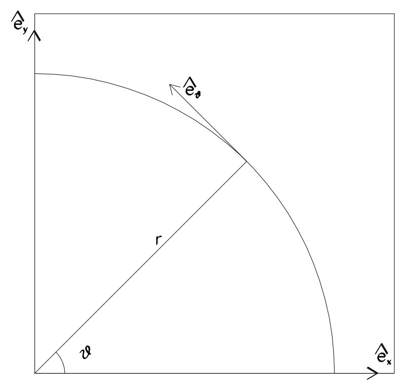
2B21 Mathematical Methods in Physics & Astronomy

Suggested Solutions for Problem Sheet M10 (2003–2004)

1. Stokes' theorem states that

$$\int_S curl\underline{A} \cdot \hat{n}\; dS = \int_\gamma \underline{A} \cdot d\underline{r}\;,$$

where the closed contour γ is along the boundary of the surface S, $d\underline{r}$ is a line element along γ, and $\hat{n}$ is a unit vector normal to S whose direction is fixed by the motion of a right-handed screw rotated in the direction of γ. [*No marks given here for this statement.*]



By simple trigonometry on the figure, we see immediately that

$$\underline{\hat{e}}_\theta = -\sin\theta\,\underline{\hat{e}}_x + \cos\theta\,\underline{\hat{e}}_y\;. \qquad \textbf{[2]}$$

For the specific problem, we are given that

$$\underline{W} = (x+y)\,\underline{\hat{e}}_x + xy^2\,\underline{\hat{e}}_y + x^2\,\underline{\hat{e}}_z\;.$$

Along (a) we have

$$I_a = \int_0^1 \underline{W} \cdot \underline{ds} = \int_0^1 x\,dx = \tfrac{1}{2}x^2\Big|_0^1 = \tfrac{1}{2}\;. \qquad \textbf{[2]}$$

On the circle of radius 1, the infinitesimal length element is

$$\underline{ds} = \underline{\hat{e}}_\theta\,d\theta = (-\sin\theta\,\underline{\hat{e}}_x + \cos\theta\,\underline{\hat{e}}_y)\,d\theta\;, \qquad \textbf{[1]}$$

so that

$$\begin{aligned} I_b &= \int_0^{\pi/2} [(\cos\theta+\sin\theta)\,\underline{\hat{e}}_x + \cos\theta\sin^2\theta\,\underline{\hat{e}}_y] \cdot (-\sin\theta\,\underline{\hat{e}}_x + \cos\theta\,\underline{\hat{e}}_y)\,d\theta \\ &= \int_0^{\pi/2} [-\sin\theta\cos\theta - \sin^2\theta + \sin^2\theta\cos^2\theta]\,d\theta\;. \qquad \textbf{[2]} \end{aligned}$$

Now

$$\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin 2\theta\, d\theta = -\left[\frac{1}{4}\cos 2\theta\right]_0^{\pi/2} = \frac{1}{2}\,.$$

$$\int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} [1-\cos 2\theta]\, d\theta = \frac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{4}\,.$$

$$\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta = \frac{1}{4}\int_0^{\pi/2} \sin^2 2\theta\, d\theta = \frac{1}{8}\int_0^{\pi/2} [1-\cos 4\theta]\, d\theta = \frac{\pi}{16}\,.$$

Hence

$$I_b = -\frac{1}{2} - \frac{\pi}{4} + \frac{\pi}{16} = -\frac{1}{2} - \frac{3\pi}{16}\,.$$ **[3]**

The final integral is much simpler because $W_y = 0$ on the last leg, which means that $I_c = 0$. **[1]**

Putting the terms together,

$$I = I_a + I_b + I_c = -\frac{3\pi}{16}\,.$$ **[1]**

To check Stokes' theorem, we must first evaluate

$$\nabla \times \underline{W} = \begin{vmatrix} \hat{\underline{e}}_x & \hat{\underline{e}}_y & \hat{\underline{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & xy^2 & x^2 \end{vmatrix} = -2x\hat{\underline{e}}_y + (y^2-1)\hat{\underline{e}}_z\,.$$ **[2]**

The normal to the surface is in the <u>positive</u> z-direction following the Stokes' theorem definition. Thus

$$\int (\nabla\times\underline{W})\cdot\underline{dS} = \int_0^1 r\,dr\int_0^{\pi/2}(y^2-1)\,d\theta = \int_0^1 r\,dr\int_0^{\pi/2}(r^2\sin^2\theta - 1)\,d\theta$$ **[1]**

$$= \int_0^1 r\,dr\int_0^{\pi/2}\left[\tfrac{1}{2}r^2(1-\cos 2\theta) - 1\right]\,d\theta = \int_0^1 r\,dr\left[\tfrac{1}{2}r^2\theta - \tfrac{1}{4}r^2\sin 2\theta - \theta\right]$$

$$= \frac{\pi}{2}\int_0^1 r\,dr[\tfrac{1}{2}r^2 - 1] = \frac{\pi}{2}\left[\frac{r^4}{8} - \frac{r^2}{4}\right]_0^1 = -\frac{3\pi}{16}\,.$$ **[3]**

Fortunately this agrees with the result of the line integral and so Stokes' theorem is valid in this case.

2. On the surface $x = 0$, the outward normal $\hat{n} = -\underline{\hat{e}}_x$, and $\underline{F} \cdot \hat{n} = -z$. Now, integrating over the quadrant,

$$I_x = \int_0^1 (-z)\, dz \int_0^{\sqrt{1-z^2}} dy = -\int_0^1 z\sqrt{1-z^2}\, dz = \frac{1}{3}(1-z^2)^{3/2}\Big|_0^1 = -\frac{1}{3} \cdot \qquad \textbf{[3]}$$

On $z = 0$ we get the same result $I_z = I_x$, whereas along $y = 0$ the flux I_y vanishes. **[1]**

On the curved surface,

$$\hat{n} = \sin\theta\cos\phi\, \underline{\hat{e}}_x + \sin\theta\sin\phi\, \underline{\hat{e}}_y + \cos\theta\, \underline{\hat{e}}_z\ ,$$

and

$$\underline{F} = \cos\theta\, \underline{\hat{e}}_x + \sin\theta\sin\phi\, \underline{\hat{e}}_y + \sin\theta\cos\phi\, \underline{\hat{e}}_z\ .$$

Hence

$$\underline{F} \cdot \hat{n} = 2\sin\theta\cos\theta\cos\phi + \sin^2\theta\sin^2\phi\ . \qquad \textbf{[2]}$$

The flux through the curved surface

$$\begin{aligned} I_s &= \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{\pi/2} d\phi [2\sin\theta\cos\theta\cos\phi + \sin^2\theta\sin^2\phi] \\ &= \int_0^{\pi/2} \sin\theta\, d\theta\, [2\sin\theta\cos\theta + \tfrac{\pi}{4}\sin^2\theta] \\ &= \left[\frac{2}{3}\sin^3\theta\right]_0^{\pi/2} - \tfrac{\pi}{4}\left[\cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \tfrac{2}{3} + \tfrac{\pi}{6}\ . \end{aligned} \qquad \textbf{[3]}$$

The total flux

$$I = I_x + I_y + I_z + I_s = \tfrac{\pi}{6}\ . \qquad \textbf{[1]}$$

Now for the easy bit! The divergence of the vector

$$\nabla \cdot \underline{F} = 1\ . \qquad \textbf{[2]}$$

Integrating this over the volume gives $\frac{1}{8}$ of the volume of the unit sphere, *viz* $\frac{1}{8}\frac{4\pi}{3} = \frac{\pi}{6}$, as before, but with only 5% of the work. **[2]**

NOTE The question should have specified the radius of the sphere by giving $x^2 + y^2 + z^2 = 1$. Any other radius chosen in answering the question would just scale the result. The marker therefore has to be sympathetic to all attempts to compensate for this error.