University College London

Department of Physics and Astronomy

2B21 Mathematical Methods in Physics & Astronomy

Suggested Solutions for Problem Sheet M9 (2003–2004)

1. For $f = x^2 + y^2 - 2z^2$, in Cartesian coordinates

$$\underline{\nabla} f = 2(x\,\underline{\hat{e}}_x + y\,\underline{\hat{e}}_y - 2z\,\underline{\hat{e}}_z)\,. \qquad \textbf{[1]}$$

In polar coordinates, $f = r^2(\sin^2\theta - 2\cos^2\theta) = r^2(1 - 3\cos^2\theta)$, so that, using the above formula,

$$\underline{\nabla} f = 2r(1 - 3\cos^2\theta)\,\underline{\hat{e}}_r + 6r\sin\theta\cos\theta\,\underline{\hat{e}}_\theta\,. \qquad \textbf{[3]}$$

To show that the two vectors are identically equal, the student has either to work geometrically or use the relation given between the basis vectors in the two coordinate systems:

$$\begin{aligned}
\underline{\hat{e}}_r &= \sin\theta\cos\phi\,\underline{\hat{e}}_x + \sin\theta\sin\phi\,\underline{\hat{e}}_y + \cos\theta\,\underline{\hat{e}}_z\,,\\
\underline{\hat{e}}_\theta &= \cos\theta\cos\phi\,\underline{\hat{e}}_x + \cos\theta\sin\phi\,\underline{\hat{e}}_y - \sin\theta\,\underline{\hat{e}}_z\,,\\
\underline{\hat{e}}_\phi &= -\sin\phi\,\underline{\hat{e}}_x + \cos\phi\,\underline{\hat{e}}_y\,,
\end{aligned}$$

Now

$$\begin{aligned}
&2r(1 - 3\cos^2\theta)\,\underline{\hat{e}}_r + 6r\sin\theta\cos\theta\,\underline{\hat{e}}_\theta\\
&\quad= 2r(1 - 3\cos^2\theta)(\sin\theta\cos\phi\,\underline{\hat{e}}_x + \sin\theta\sin\phi\,\underline{\hat{e}}_y + \cos\theta\,\underline{\hat{e}}_z)\\
&\qquad +6r\sin\theta\cos\theta(\cos\theta\cos\phi\,\underline{\hat{e}}_x + \cos\theta\sin\phi\,\underline{\hat{e}}_y - \sin\theta\,\underline{\hat{e}}_z)\\
&\quad= 2r\sin\theta\cos\phi(1 - 3\cos^2\theta + 3\cos^2\theta)\underline{\hat{e}}_x\\
&\qquad +2r\sin\theta\sin\phi(1 - 3\cos^2\theta + 3\cos^2\theta)\underline{\hat{e}}_y\\
&\qquad +[2r(1 - 3\cos^2\theta)\cos\theta - 6r\sin^2\theta\cos\theta]\,\underline{\hat{e}}_z\\
&\quad= 2r\sin\theta\,\underline{\hat{e}}_x + 2r\sin\theta\sin\phi\,\underline{\hat{e}}_y - 2r\cos\theta\,\underline{\hat{e}}_z = 2x\,\underline{\hat{e}}_x + 2y\,\underline{\hat{e}}_y - 4z\,\underline{\hat{e}}_z\,,
\end{aligned}$$

which is the same answer as in Cartesians. **[4]**

2. We have to verify

$$\nabla \times (\underline{A} \times \underline{B}) = \underline{A}(\nabla \cdot \underline{B}) - (\underline{A} \cdot \nabla)\underline{B} + (\underline{B} \cdot \nabla)\underline{A} - \underline{B}(\nabla \cdot \underline{A})$$

for the vector fields

$$\underline{A} = x\,\hat{\underline{e}}_x + y\,\hat{\underline{e}}_y + z\,\hat{\underline{e}}_z\,,$$
$$\underline{B} = -y\,\hat{\underline{e}}_x + x\,\hat{\underline{e}}_y\,.$$

$$\underline{A} \times \underline{B} = \begin{vmatrix} \hat{\underline{e}}_x & \hat{\underline{e}}_y & \hat{\underline{e}}_z \\ x & y & z \\ -y & x & 0 \end{vmatrix} = -xz\,\hat{\underline{e}}_x - yz\,\hat{\underline{e}}_y + (x^2+y^2)\,\hat{\underline{e}}_z\,.$$ **[2]**

Hence

$$\nabla\times(\underline{A}\times\underline{B}) = \begin{vmatrix} \hat{\underline{e}}_x & \hat{\underline{e}}_y & \hat{\underline{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -xz & -yz & x^2+y^2 \end{vmatrix} = (2y+y)\hat{\underline{e}}_x - (2x+x)\hat{\underline{e}}_y + (2z)\hat{\underline{e}}_z = 3y\,\hat{\underline{e}}_x - 3x\,\hat{\underline{e}}_y.$$ **[3]**

However,

$$\nabla \cdot \underline{B} = 0 \quad \text{and} \quad \nabla \cdot \underline{A} = 3\,,$$

so that

$$\underline{A}(\nabla \cdot \underline{B}) = 0,$$
$$-\underline{B}(\nabla \cdot \underline{A}) = 3(y\,\hat{\underline{e}}_x - x\,\hat{\underline{e}}_y)\,.$$

The other two terms are

$$-(\underline{A} \cdot \nabla)\underline{B} = -\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)(-y\,\hat{\underline{e}}_x + x\,\hat{\underline{e}}_y) = y\,\hat{\underline{e}}_x - x\,\hat{\underline{e}}_y\,,$$

$$(\underline{B} \cdot \nabla)\underline{A} = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)(x\,\hat{\underline{e}}_x + y\,\hat{\underline{e}}_y + z\,\hat{\underline{e}}_z) = -y\,\hat{\underline{e}}_x + x\,\hat{\underline{e}}_y\,.$$ **[3]**

Thus

$$\underline{A}(\nabla \cdot \underline{B}) - (\underline{A} \cdot \nabla)\underline{B} + (\underline{B} \cdot \nabla)\underline{A} - \underline{B}(\nabla \cdot \underline{A})$$
$$= 3(y\,\hat{\underline{e}}_x - x\,\hat{\underline{e}}_y) + (-y\,\hat{\underline{e}}_x + x\,\hat{\underline{e}}_y) + (y\,\hat{\underline{e}}_x - x\,\hat{\underline{e}}_y) = 3(y\,\hat{\underline{e}}_x - x\,\hat{\underline{e}}_y)\,.$$ **[2]**

This verifies the identity.

3. In spherical polar coordinates,

$$\underline{A} = r\,\underline{\hat{e}}_r\,,$$

$$\underline{B} = r\sin\theta\,\underline{\hat{e}}_\phi\,,$$

so that

$$\underline{F} = \underline{A}\times\underline{B} = \begin{vmatrix} \underline{\hat{e}}_r & \underline{\hat{e}}_\theta & \underline{\hat{e}}_\phi \\ r & 0 & 0 \\ 0 & 0 & r\sin\theta \end{vmatrix} = -r^2\sin\theta\,\underline{\hat{e}}_\theta\,. \qquad \textbf{[2]}$$

The expression for curl is given in the hand-out:

$$(\nabla\times\underline{F})_r = \frac{1}{r\,\sin\theta}\left\{\frac{\partial}{\partial\theta}\{\sin\theta\,F_\phi(r,\theta,\phi)\} - \frac{\partial}{\partial\phi}F_\theta(r,\theta,\phi)\right\}.$$

$$(\nabla\times\underline{F})_\theta = \frac{1}{r\,\sin\theta}\left\{\frac{\partial}{\partial\phi}F_r(r,\theta,\phi) - \sin\theta\,\frac{\partial}{\partial r}\,\{r\,F_\phi(r,\theta,\phi)\}\right\},$$

$$(\nabla\times\underline{F})_\phi = \frac{1}{r}\left\{\frac{\partial}{\partial r}\,\{r\,F_\theta(r,\theta,\phi)\} - \frac{\partial}{\partial\theta}F_r(r,\theta,\phi)\right\}.$$

Since only the θ component of $\underline{F}$ is non-zero, this reduces to

$$(\nabla\times\underline{F})_r = \frac{1}{r\,\sin\theta}\left\{\frac{\partial}{\partial\phi}\,(r^2\sin\theta)\right\},$$

$$(\nabla\times\underline{F})_\theta = 0\,,$$

$$(\nabla\times\underline{F})_\phi = -\frac{1}{r}\left\{\frac{\partial}{\partial r}\,\left\{r^3\sin\theta\right\}\right\}.$$

The differentiation with respect to ϕ gives nothing, and so we are left just with

$$\nabla\times(\underline{A}\times\underline{B}) = -3r\sin\theta\,\underline{\hat{e}}_\phi\,, \qquad \textbf{[3]}$$

which does agree with the result of question 1.

Since $\underline{A}$ only has a radial component,

$$\underline{B}(\nabla\cdot\underline{A}) = r\sin\theta\,\underline{\hat{e}}_\phi\,\frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^3\right) = 3r\sin\theta\,\underline{\hat{e}}_\phi\,, \qquad \textbf{[1]}$$

and $\underline{B}$ only an azimuthal component,

$$\underline{A}(\nabla\cdot\underline{B}) = r\,\underline{\hat{e}}_r\,\frac{1}{r\sin\theta}\,\frac{\partial}{\partial\phi}\left(r^2\sin\theta\right) = \underline{0}\,. \qquad \textbf{[1]}$$

Now, using the expression for the gradient,

$$(\underline{A} \cdot \nabla)\, \underline{B} = r\, \frac{\partial}{\partial r}(r \sin\theta\, \underline{\hat{e}}_\phi) = r \sin\theta\, \underline{\hat{e}}_\phi\ . \qquad \textbf{[1]}$$

$$(\underline{B} \cdot \nabla)\, \underline{A} = r \sin\theta\, \frac{1}{r \sin\theta}\, \frac{\partial}{\partial \phi}(r\, \underline{\hat{e}}_r) = \frac{\partial}{\partial \phi}(r\, \underline{\hat{e}}_r)\ .$$

Now the differentiation of r with respect to ϕ clearly vanishes, but the basis vector itself depends upon ϕ, so that

$$\frac{\partial}{\partial \phi}\, \underline{\hat{e}}_r = -\sin\theta \sin\phi\, \underline{\hat{e}}_x + \sin\theta \cos\phi\, \underline{\hat{e}}_y = \underline{\hat{e}}_\phi\ ,$$

which means that

$$(\underline{B} \cdot \nabla)\, \underline{A} = r \sin\theta\, \underline{\hat{e}}_\phi\ . \qquad \textbf{[3]}$$

Putting everything together,

$$\begin{aligned} &\underline{A}(\nabla \cdot \underline{B}) - (\underline{A} \cdot \nabla)\underline{B} + (\underline{B} \cdot \nabla)\underline{A} - \underline{B}(\nabla \cdot \underline{A}) \\ &= 0 - 3r \sin\theta\, \underline{\hat{e}}_\phi + r \sin\theta\, \underline{\hat{e}}_\phi - r \sin\theta\, \underline{\hat{e}}_\phi = -3r \sin\theta\, \underline{\hat{e}}_\phi\ , \qquad \textbf{[1]} \end{aligned}$$

which verifies the identity in spherical polar coordinates.