University College London

Department of Physics and Astronomy

2B21 Mathematical Methods in Physics & Astronomy

Suggested Solutions for Problem Sheet M7 (2003–2004)

1. We are given that

$$\int_{-1}^{+1} P_m(x)P_n(x)\,dx = \delta_{nm}\,.$$

Starting with the cases where $n \neq m$, since $P_1(x)$ is odd and $P_0(x)$ and $P_2(x)$ are even, it follows immediately that

$$\int_{-1}^{+1} P_0(x)P_1(x)\,dx = \int_{-1}^{+1} P_2(x)P_1(x)\,dx = 0\,. \qquad \textbf{[1]}$$

Furthermore,

$$\int_{-1}^{+1} P_2(x)P_0(x)\,dx = \tfrac{1}{2}\int_{-1}^{+1}(3x^2-1)\,dx = \tfrac{1}{2}\left[x^3 - x\right]_{-1}^{+1} = 0\,. \qquad \textbf{[2]}$$

For $n = m = 0$,

$$\int_{-1}^{+1} \left[P_0(x)\right]^2\,dx = \int_{-1}^{+1} dx = 2\,. \qquad \text{[1]}$$

For $n = m = 1$,

$$\int_{-1}^{+1} \left[P_1(x)\right]^2\,dx = \int_{-1}^{+1} x^2\,dx = \tfrac{2}{3}\,. \qquad \text{[1]}$$

For $n = m = 2$,

$$\int_{-1}^{+1} \left[P_2(x)\right]^2\,dx = \tfrac{1}{4}\int_{-1}^{+1}(9x^4 - 6x^2 + 1)\,dx = \tfrac{1}{4}\left[\tfrac{9}{5}x^5 - 2x^3 + x\right]_{-1}^{+1} = \tfrac{2}{5}\,. \qquad \text{[1]}$$

If $P_3(x) = a[x^3 + bx^2 + cx + d]$, orthogonality with $P_0(x)$ requires that

$$\int_{-1}^{+1}[x^3 + bx^2 + cx + d]\,dx = \tfrac{2}{3}b + 2d = 0\,. \qquad \text{[1]}$$

Similarly, orthogonality with $P_1(x)$ necessitates

$$\int_{-1}^{+1}[x^4 + bx^3 + cx^2 + dx]\,dx = \tfrac{2}{5} + \tfrac{2}{3}c = 0.\,. \qquad \text{[1]}$$

Finally, imposing orthogonality with respect to $P_2(x)$ means that

$$\tfrac{3}{2}\int_{-1}^{+1}[x^5+bx^4+cx^3+dx^2]\,dx+\tfrac{1}{2}\int_{-1}^{+1}[x^3+bx^2+cx+d]\,dx=0\,.$$

Now the second integral vanishes because of the orthogonality to $P_0(x)$. Hence

$$\tfrac{3}{5}b+\tfrac{1}{2}d=0\,. \qquad \textbf{[1]}$$

This result is incompatible with the previous relation obtained between b and d. Hence $b=d=0$. Students might get this result by claiming that $P_3(x)$ is an odd function. Though true, this was not given in the question and would only receive a maximum credit of **three** marks.

Using the result for c, the Legendre polynomial reduces to

$$P_3(x)=a\left[x^3-\tfrac{3}{5}x\right]\,. \qquad \textbf{[1]}$$

The easiest way of determining the value of a is from the condition that the Legendre polynomials are normalised by $P_n(x=1)=1$. Therefore

$$1=a\left[1-\tfrac{3}{5}\right]=\tfrac{2}{5}a\,,$$

and $a=\frac{5}{2}$. **[3]**

<u>Alternatively</u>, For $n=m=2$,

$$\int_{-1}^{+1}[P_3(x)]^2\,dx=a^2\int_{-1}^{+1}\left(x^6-\tfrac{6}{5}x^4+\tfrac{9}{25}x^2\right)\,dx$$

$$=a^2\left[\tfrac{1}{7}x^7-\tfrac{6}{25}x^5+\tfrac{3}{25}x^3\right]_{-1}^{+1}=\tfrac{8}{175}a^2=\tfrac{2}{7}\,.$$

Hence $a^2=25/4$ and we find once more that $a=\frac{5}{2}$. **[3]**

2. Expanding the left hand side of

$$g(x,t)=\frac{\exp\left(-xt/(1-t)\right)}{1-t}=\sum_{n=0}^{\infty}L_n(x)\,t^n$$

in powers of t gives

$$g(x,t)=(1+t+t^2)\,\exp(-xt(1+t))+O(t^3)\approx(1+t+t^2)(1-xt-xt^2+\tfrac{1}{2}x^2t^2) \qquad \textbf{[1]}$$

$$=1+t(1-x)+t^2(\tfrac{1}{2}x^2-2x+1)+O(t^3)\,.$$

Hence $L_0(x)=1$, $L_1(x)=1-x$, **[1]**

and $L_2(x)=\frac{1}{2}(x^2-4x+2)$. **[1]**

Differentiating the generating function with respect to x gives

$$\frac{\partial g(x,t)}{\partial x} = -\left(\frac{t}{1-t}\right) g(x,t) = \sum_{n=0}^{\infty} L'_n(x)\, t^n\,. \qquad \textbf{[1]}$$

Expanding $g(x,t)$ as a power series and multiplying through by $(1-t)$ results in

$$t\sum_{n=0}^{\infty} L_n(x)\, t^n = (1-t)\sum_{n=0}^{\infty} L'_n(x)\, t^n\,.$$

$$\sum_{n=0}^{\infty} L_n(x)\, t^{n+1} = \sum_{n=0}^{\infty} L'_n(x)\, t^n - \sum_{n=0}^{\infty} L'_n(x)\, t^{n+1}\,. \qquad \textbf{[1]}$$

Now change the summation index $n \to n+1$ in the first term on the right hand side and compare the coefficients of t^{n+1};

$$L_n(x) = L'_{n+1}(x) - L'_n(x)\,. \qquad \textbf{[2]}$$

Now

$$g(x,t) = \frac{\exp\left(-xt/(1-t)\right)}{1-t} = \sum_{n=0}^{\infty} L_n(x)\, t^n$$

$$g(x,u) = \frac{\exp\left(-xu/(1-u)\right)}{1-u} = \sum_{n=0}^{\infty} L_n(x)\, u^n \qquad \textbf{[1]}$$

$$I = \int_0^{\infty} e^{-x} g(x,t)\, g(x,u)\, dx = \frac{1}{(1-t)(1-u)}\int_0^{\infty} \exp\left[-x\left(1+\frac{t}{1-t}+\frac{u}{1-u}\right)\right] dx$$

$$= \frac{1}{(1-t)(1-u)}\int_0^{\infty} \exp\left[-x\left(\frac{1-ut}{(1-t)(1-u)}\right)\right] dx = \frac{1}{1-ut} = \sum_{n=0}^{\infty} u^n t^n\,. \qquad \textbf{[2]}$$

Using the expansion formulae, the integral must also be given by

$$I = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^n t^m \int_0^{\infty} e^{-x} L_n(x)\, L_m(x)\, dx\,. \qquad \textbf{[1]}$$

On the left hand side the power of u is always equal to the power of t. Hence only $n = m$ is non-vanishing on the RHS so that the required integral is proportional to δ_{nm}. **[1]**

Then, comparing powers of $u^n t^n$ on both sides, this shows that

$$\int_0^{\infty} e^{-x} L_n(x)\, L_n(x)\, dx = 1\,. \qquad \textbf{[1]}$$

This gives the normalisation integral for the Coulomb wave functions that is of use in the 2B22 course.