University College London

Department of Physics and Astronomy

2B21 Mathematical Methods in Physics & Astronomy

Suggested Solutions for Problem Sheet M6 (2003–2004)

1. Inserting the *ansatz* $u(x,y) = X(x) \times Y(y)$ into the partial differential equation

$$x\,\frac{\partial u}{\partial x} - y\,\frac{\partial u}{\partial y} = u\;.$$

leads to

$$\frac{x}{X}\,\frac{dX}{dx} = 1 + \frac{y}{Y}\,\frac{dY}{dy}\;. \qquad \textbf{[2]}$$

Since the left hand side is a function of x and the right of y, this means that they must both equal some separation constant α. We then obtain two ordinary differential equations

$$\begin{aligned} \frac{dX}{dx} &= \alpha\frac{1}{x}\,X\;, \\ \frac{dY}{dy} &= (\alpha-1)\frac{1}{y}\,Y\;, \end{aligned} \qquad \textbf{[1]}$$

where α is as yet arbitrary.

The first equation can be written as

$$\int \frac{dX}{X} = \alpha \int \frac{dx}{x}\;,$$

which may be integrated to give

$$\ell n(X/A_\alpha) = \alpha\,\ell n(x)\;,$$

where A_α is an arbitrary integration constant.

Taking exponential of both sides,

$$X = A_\alpha\,x^\alpha\;.$$

The y equation is exactly the same except that α is replaced by $\alpha - 1$ so that the solution is

$$Y = B_\alpha\,y^{\alpha-1}\;.$$

The corresponding solution for $u(x,y)$ is

$$u(x,y) = C_\alpha\,x^\alpha y^{\alpha-1} = C_\alpha\,x(xy)^{\alpha-1}\;,$$

where $C_\alpha = A_\alpha B_\alpha$ is the combined integration constant.

The most general solution to the differential equation is a linear superposition of such solutions:

$$u(x,y) = \sum_\alpha C_\alpha \, x^\alpha y^{\alpha-1}\,, \qquad \textbf{[5]}$$

where the sum may actually be an integral over continuous values of α (though students would not be expected to stress this point).

On the line $y = x$ we have

$$u(x,x) = \sum_\alpha C_\alpha \, x^{2\alpha-1} = x + x^3\,.$$

Hence $C_1 = 1$ and $C_2 = 1$ with all the other coefficients vanishing. Hence the specific solution is

$$u(x,y) = x + x^2 y\,. \qquad \textbf{[2]}$$

2. Assume that a solution of the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial}{\partial r}V(r,\theta)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}V(r,\theta) = 0$$

exists in the form

$$V(r,\theta) = R(r)\;\times\;\Theta(\theta)\,.$$

Straightforward differentiating then leads to

$$\frac{\Theta}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} = 0\,.$$

Dividing through by $R\Theta$, multiplying up by r^2, and taking one term over to the other side, the equation separates as

$$r\,\frac{1}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}\,. \qquad \textbf{[1]}$$

Since the LHS is a function only of r and the RHS of θ, they must both be equal to some separation constant which we put equal to $+n^2$. This then yields two ordinary differential equations

$$r\,\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2\,R\,,$$

$$\frac{d^2\Theta}{d\theta^2} + n^2\,\Theta = 0\,. \qquad \textbf{[1]}$$

The most general solution of the θ equation is

$$\Theta = C_n\cos n\theta + D_n\sin n\theta\,. \qquad \textbf{[1]}$$

Now, in order that $V(r,\theta)$ be single-valued as $\theta \to \theta + 2\pi$, n must be an integer. **[1]**
Turning to the R equation, if $n = 0$ we must solve

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0 \quad \Rightarrow \quad r\frac{dR}{dr} = B \quad \Rightarrow \quad R = A + B\,\ell n\, r\,. \qquad \textbf{[1]}$$

Otherwise, look for a solution of the form $R = r^\alpha$. This is possible if

$$\alpha^2\, r^\alpha = n^2\, r^\alpha \quad \Rightarrow \quad \alpha = \pm n\,.$$

The general solution then is

$$R = C\, r^n + \frac{D}{r^n}\,. \qquad \textbf{[1]}$$

Putting everything together,

$$V(r,\theta) = A + B\,\ell n\, r + \sum_{n=1}^{\infty}\left(C_n r^n + \frac{D_n}{r^n}\right)\left(E_n \cos n\theta + F_n \sin n\theta\right)\,. \qquad \textbf{[1]}$$

In order to avoid divergent terms in the region $r \leq a$, we must have $B = 0$ and $D_n = 0$, so that

$$V(r,\theta) = A + \sum_{n=1}^{\infty} C_n r^n\,\left(E_n \cos n\theta + F_n \sin n\theta\right)\,. \qquad \textbf{[1]}$$

Now at $r = a$ the potential is of the form $V_0 \cos\theta$, so that $A = 0$, $F_n = 0$, and $E_n = 0$ unless $n = 1$ when $C_1 E_1 = V_0/a$. Hence for $r \leq a$,

$$V(r,\theta) = V_0\left(\frac{r}{a}\right)\,. \qquad \textbf{[1]}$$

Similarly in the region where $r \geq a$ we require $B = 0$ and $C_n = 0$. Matching the boundary conditions at $r = a$ then leads to

$$V(r,\theta) = V_0\left(\frac{a}{r}\right), \quad (r \geq a)\,. \qquad \textbf{[1]}$$