PHAS1245: Problem Sheet 9 - Solutions

1. (a) We have

$$S = \sum_{n=0}^{500} (1000 + 2n)$$

(including 1000 for $n = 0$ and 2000 for $n = 500$, but full marks to those who would not include either of those, as long as they specify it). Then, by writing the sum in reverse order and summing the two, as we did in the lectures, we get

$$2S = 501(2000 + 2 \times 500) \Rightarrow S = 501 \times 1500 = 751,500\,.$$

(b) The 1st 1000 will be multiplied by 1.05 every year for 25 years, the 2nd 1000 for 24 years etc... and the 25th 1000 will be multiplied by 1.05 only once. So, this sum can be written as

$$S = \sum_{n=1}^{25} 1000 \times 1.05^n\,,$$

i.e it is a geometric series. By multiplying the above by 1.05 and subtracting, as we did in the lectures for any geometric series, we get

$$(1 - 1.05)S = 1000(1.05 - 1.05^{26}) \Rightarrow S = 1000\frac{1.05 - 1.05^{26}}{1 - 1.05} = 51,113.45\,.$$

2. We have

$$S_{N+1} = a(N+1)^3 + b(N+1)^2 + c(N+1) = aN^3 + (3a+b)N^2 + (3a+2b+c)N + (a+b+c)\,,$$

but also

$$S_{N+1} = S_N + (N+1)^2 = aN^3 + bN^2 + cN + (N+1)^2 = aN^3 + (b+1)N^2 + (c+2)N + 1\,.$$

Therefore

$$aN^3 + (3a+b)N^2 + (3a+2b+c)N + (a+b+c) = aN^3 + (b+1)N^2 + (c+2)N + 1$$

and for this to be true independently of N, the coefficients of the same powers of N must be equal. Hence

$$3a + b = b + 1 \Rightarrow a = \frac{1}{3}\,, \quad 3a + 2b + c = c + 2 \Rightarrow b = \frac{1}{2}\,, \quad a + b + c = 1 \Rightarrow c = \frac{1}{6}\,.$$

3. Any correct expansion (around whichever x_0, even $x_0 = 0$) will receive full marks.

   Taking $f(x) = ln(1+x)$ then

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}, \quad f''''(x) = -\frac{2\times 3}{(1+x)^4},$$

   so, in general, the nth derivative at $x = x_0$ is

$$f^{(n)}(x_0) = (-1)^{n+1}\frac{(n-1)!}{(1+x_0)^n}$$

   and the Taylor expansion around x_0 is:

$$\ln(1+x) = \ln(1+x_0) + \sum_{n=1}^{\infty}\left[(-1)^{n+1}\frac{(n-1)!}{n!(1+x_0)^n}(x-x_0)^n\right]$$

$$\Rightarrow \ln(1+x) = \ln(1+x_0) + \sum_{n=1}^{\infty}\left[(-1)^{n+1}\frac{(x-x_0)^n}{n(1+x_0)^n}\right].$$

   (note that clearly this is most useful when $x_0 = 0$, since then $\ln(1+x_0) = \ln(1) = 0$, otherwise you still need a calculator to get $\ln(1+x_0)$!).

   For the second function, we can write $\ln[(1+x)/(1-x)] = \ln(1+x) - \ln(1-x)$. We have the expansion of the first term, so we need only the second one. The derivatives for $\ln(1-x)$ are almost the same as above, only there is an additional minus sign that comes up every time because of $-x$ and this always cancels the negative sign that comes about from the negative powers. So

$$\frac{d^n}{dx^n}(\ln(1-x))\bigg|_{x=x_0} = -\frac{(n-1)!}{(1-x_0)^n}$$

   and

$$\ln\left(\frac{1+x}{1-x}\right) = \ln\left(\frac{1+x_0}{1-x_0}\right) + \sum_{n=1}^{\infty}\left[(-1)^{n+1}\frac{(x-x_0)^n}{n(1+x_0)^n}\right] - \sum_{n=1}^{\infty}\left[-\frac{(x-x_0)^n}{n(1-x_0)^n}\right].$$

   For a McLaurin series $x_0 = 0$, hence

$$\Rightarrow \ln\left(\frac{1+x}{1-x}\right) = \sum_{n=1}^{\infty}\left[(-1)^{n+1}\frac{x^n}{n}\right] + \sum_{n=1}^{\infty}\left[\frac{x^n}{n}\right].$$

   This means that the even terms in the two sums cancel and we are left only with the odd terms twice.

$$\Rightarrow \ln\left(\frac{1+x}{1-x}\right) = 2\sum_{n=0}^{\infty}\left[\frac{x^{2n+1}}{2n+1}\right].$$