PHAS1245: Problem Sheet 5 - Solutions

1. Since $\vec{A} = 3\hat{\imath} + 5\hat{\jmath} - 7\hat{k}$ and $\vec{B} = 2\hat{\imath} + 7\hat{\jmath} + \hat{k}$

$$\vec{A} + \vec{B} = 5\hat{\imath} + 12\hat{\jmath} - 6\hat{k} \qquad \vec{A} - \vec{B} = \hat{\imath} - 2\hat{\jmath} - 8\hat{k}\,,$$

$$|\vec{A}| = \sqrt{83} \qquad |\vec{B}| = \sqrt{54} \qquad \vec{A}\cdot\vec{B} = 34 \qquad cos\theta = 0.5078\,.$$

2. The vector product of two vectors gives a vector perpendicular to both i.e.

$$\vec{C} = (\hat{\imath} + \hat{\jmath} - \hat{k}) \times (2\hat{\imath} - \hat{\jmath} + 3\hat{k}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 2\hat{\imath} - 5\hat{\jmath} - 3\hat{k}$$

and

$$\hat{C} = (2\hat{\imath} - 5\hat{\jmath} - 3\hat{k})/\sqrt{38}\,.$$

3. Since $\vec{a} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$, $\vec{b} = -\hat{\imath} + \hat{k}$ and $\vec{c} = 3\hat{\imath} + \hat{\jmath} - \hat{k}$, we get $\vec{a}\cdot\vec{b} = 0$ and

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 2\hat{\imath} - 2\hat{\jmath} + 2\hat{k}\,.$$

For the triple products we have

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 3 & 1 & -1 \end{vmatrix} = 2$$

and

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b}) = -4\hat{\imath} + 4\hat{k}\ .$$

(There are other ways of calculating the last two results eg first find $\vec{b}\times\vec{c}$ and then calculate $\vec{a}\cdot(\vec{b}\times\vec{c})$ or $\vec{a}\times(\vec{b}\times\vec{c})$ as appropriate.)

4. This can either be done by resolving $\vec{A}$ into a vector in the direction of $\hat{n}$, (this is the vector $|\vec{A}|cos\theta\hat{n}$ or $(\vec{A}\cdot\hat{n})\hat{n}$) and into the vector $|\vec{A}|sin\theta\hat{p}$ where $\hat{p}$ is perpendicular to $\hat{n}$ and lies in the plane of $\vec{A}$ and $\hat{n}$. The second component vector is $(\hat{n}\times\vec{A})\times\hat{n}$.

   Or it can be done by expanding the triple vector product term on the right hand side $((\hat{n}\times\vec{A})\times\hat{n})$ as was done in the previous question.

5. If the position vectors are $\vec{A} = 3\hat{\imath} - 4\hat{\jmath} + 0\hat{k}$ and $\vec{B} = -2\hat{\imath} + 1\hat{\jmath} + 0\hat{k}$ then $|A| = \sqrt{9+16+0} = 5$ and $|B| = \sqrt{4+1+0} = \sqrt{5}$. Thus $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = 5\sqrt{5}$. But also

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = -10 \Rightarrow \cos\theta = -2/\sqrt{5} \Rightarrow \theta = 153.4^0.$$

Now

$$\left(\vec{A}\times\vec{B}\right)\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -4 & 0 \\ -2 & 1 & 0 \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} - 5\hat{k}$$

This vector only has a component in the z direction. The unit vector can be taken as $(0,0,1)$, and these components are also the required direction cosines.

6. From the standard form of the equation for a plane we can conclude that both planes go through the origin (the rhs of both equations are zero), and $(1, 2, 3)$ is a vector perpendicular to the first, and $(3, 2, 1)$ perpendicular to the second, plane. Thus

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = -4\hat{\imath} + 8\hat{\jmath} - 4\hat{k}$$

determines the direction of the line of intersection. Normalising this vector we get $(-1, 2, -1)\sqrt{6}$ and these are also the required direction cosines.

The second way to solve this: We already have one point common to the two planes , namely the origin $(0, 0, 0)$. The two equations of the planes, when solved simultaneously determine two coordinates of the common point if one of the coordinates is given or chosen. For example if we multiply the second equation by 3 and eliminate z we find $x = -y/2$. So if we chose $y = 2$ then $x = -1$ and $z = -1$. Since the origin is a common point as well then the second common position vector is also the line of intersection and it is indeed the same vector found from the vector product above (modulo the factor $\sqrt{6}$). Normalising this vector we get the same direction cosines as above.