PHAS1245: Problem Sheet 4 - Solutions

1. We have
$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\tan\frac{x}{2}\cos^2\frac{x}{2}\,.$$
Also
$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow 1 + \tan^2\theta = \frac{1}{\cos^2\theta} \Rightarrow \cos^2\theta = \frac{1}{1+\tan^2\theta}$$
and by substituting the second into the first we get
$$\sin x = \frac{2t}{1+t^2}\,.$$
Now by setting $t = \tan\frac{x}{2}$ we also have
$$dx = \frac{dx}{dt}\,dt = \frac{1}{\frac{dt}{dx}}\,dt = \frac{1}{\frac{1}{2\cos^2\frac{x}{2}}}\,dt = \frac{2}{1+t^2}\,dt\,,$$
hence
$$\int \frac{1}{\sin x}\,dx = \int \frac{1+t^2}{2t}\frac{2}{1+t^2}\,dt = \int \frac{dt}{t} = \ln t = \ln\tan\frac{x}{2}$$

2. Since this is an even function:
$$\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}} \Rightarrow \int_0^{\infty} \mathrm{e}^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{4\alpha}}\,.$$
Then, differentiating both sides of the above with respect to α we get
$$\frac{d}{d\alpha}\left(\int_0^{\infty} \mathrm{e}^{-\alpha x^2}\,dx\right) = \int_0^{\infty} -x^2\mathrm{e}^{-\alpha x^2}\,dx = \frac{d}{d\alpha}\left(\frac{\sqrt{\pi}}{2}\alpha^{-\frac{1}{2}}\right) = \frac{1}{2}\left(-\frac{1}{2}\right)\sqrt{\pi}\alpha^{-\frac{3}{2}}$$
$$\Rightarrow \int_0^{\infty} x^2 e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}\,.$$

3. We require
$$\int_0^{\infty} P(v)dv = A\int_0^{\infty} v^2\mathrm{e}^{-\frac{1}{2}\frac{mv^2}{kT}}\,dv = 1\,,$$
i.e.
$$A\frac{\sqrt{\pi}}{4}\left(\frac{2kT}{m}\right)^{\frac{3}{2}} = 1\,.$$
Thus
$$A = \sqrt{\frac{2}{\pi}\left(\frac{m}{kT}\right)^3}\,.$$

4. From the definitions
$$\sinh z = \frac{1}{2}(\mathrm{e}^z - \mathrm{e}^{-z}) \qquad\qquad \cosh z = \frac{1}{2}(\mathrm{e}^z + \mathrm{e}^{-z})$$
we readily get the first: $\cosh z + \sinh z = \mathrm{e}^z$. For the second we get:
$$\sinh^2 z = \frac{1}{4}(\mathrm{e}^{2z} + \mathrm{e}^{-2z} - 2)$$

and

$$\cosh^2 z = \frac{1}{4}(\mathrm{e}^{2z} + \mathrm{e}^{-2z} + 2)\,.$$

Hence

$$\cosh^2 z - \sinh^2 z = 1\,.$$

5. Complete the square:

$$x^2 + 6x + 1 = x^2 + 6x \pm 9 + 1 = (x+3)^2 - (2\sqrt{2})^2\,.$$

We saw in the lectures that

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a}\,.$$

(**for the markers: Please note that in the lectures I made a mistake and had an extra 1/a multiplying the result. I corrected this on Monday, after a student spotted it, so please be lenient in case there is this mistake in some answers**).

Hence

$$\int \frac{1}{\sqrt{x^2 + 6x + 1}}\,dx = \int \frac{1}{\sqrt{(x+3)^2 - (2\sqrt{2})^2}}\,dx = \cosh^{-1}\frac{x+3}{2\sqrt{2}}\,.$$