PHAS1245: Problem Sheet 2 - Solutions

1.

$$\frac{d}{dx}\left(x^2e^x\right) = 2xe^x + x^2e^x = xe^x(x+2)\ .$$

2.

$$\frac{d}{dx}\ln\left(a^x + a^{-x}\right) = \frac{1}{(a^x + a^{-x})}\left\{a^x \ln a + a^{-x}(-1)\ln a\right\}$$

$$= \ln a \frac{a^x - a^{-x}}{a^x + a^{-x}}\ .$$

3.

$$\frac{d}{dx}\ln\left(x^a + x^{-a}\right) = \frac{1}{x^a + x^{-a}}\left\{ax^{a-1} + (-a)x^{-a-1}\right\}$$

$$= \frac{\left(ax^{(a-1)} - ax^{-a-1}\right)}{x^a + x^{-a}} = \frac{a\left(x^a - x^{-a}\right)}{x\left(x^a + x^{-a}\right)}\ .$$

4. If $y = x^x$ then $\ln y = x \ln x$. Now differentiate both side wrt x:

$$\frac{1}{y}\frac{dy}{dx} = \ln x + x\frac{1}{x}$$

$$\text{Thus}\quad \frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{y}\left(\ln \mathrm{x} + 1\right)$$

$$\text{and}\quad \frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{x}^{\mathrm{x}}\left(\ln \mathrm{x} + 1\right)\ .$$

5.

$$\frac{d}{dr}\left(r^2 + d^2 - 2rd\cos\theta\right)^{-\frac{1}{2}} = -\frac{1}{2}\frac{2r - 2d\cos\theta}{\left(r^2 + d^2 - 2rd\cos\theta\right)^{\frac{3}{2}}}$$

$$= -\frac{r - d\cos\theta}{\left(r^2 + d^2 - 2rd\cos\theta\right)^{\frac{3}{2}}}\ .$$

6. $y = \arcsin x$ means $x = \sin y$. Then use inverse function differentiation:

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{cosy}$$

But

$$\cos^2 y = 1 - \sin^2 y = 1 - x^2 \Rightarrow \cos y = +\sqrt{1 - x^2}$$

(choose + because the slope of arcsin is always positive). Hence

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}$$

7. The shape is a rectangle with the top side replaced by half a circle. If r is the radius of the circle and a is the vertical side of the rectangle, then the area A of the tunnel is

$$A = 2ra + \frac{1}{2}\pi r^2$$

and its perimeter S is

$$S = 2a + 2r + \pi r\,.$$

Solving the first for a and substituting in the second, we get

$$S = \frac{A}{r} + \left(2 + \frac{\pi}{2}\right) r$$

We want to minimize S wrt r, so

$$\frac{dS}{dr} = -\frac{A}{r^2} + 2 + \frac{\pi}{2} = 0 \Rightarrow A = r^2\left(2 + \frac{\pi}{2}\right)\,.$$

The second derivative is

$$\frac{d^2S}{dr^2} = \frac{2A}{r^3}\,,$$

greater than 0 for all r, so the stationary point is a minimum (as we required!). Hence, substituting A in the very first equation we find $a = r$.