PHAS1245: Problem Sheet 1 - Solutions

1.

$$\text{We have } 2x^2 - 3x - 5 = x^2 - \frac{3}{2}x - \frac{5}{2} = (x - \frac{3}{4})^2 - \frac{9}{16} - \frac{10}{4} = 0 \ .$$

$$\text{Thus } x = \frac{3 \pm 7}{4} \ .$$

[2]

2. In $2^{2x} + 3(2^x) - 4 = 0$ substitute $y = 2^x$ then

$$y^2 + 3y - 4 = (y + 4)(y - 1) = 0 \ ,$$

$$\text{i.e. } y = -4 \text{ or } y = 1 \text{ and } 2^x = -4 \text{ or } 2^x = 1 \ .$$

There are no real values of x for which $2^x = -4$. But if $2^x = 1$ then $x = 0$ is the only solution to this equation. **[6]**

3. Since

$$E = E_1 + E_2 \text{ and } sin\frac{\phi}{2} = \frac{m}{2\sqrt{E_1 E_2}} \ , 4E_1 E_2 = \frac{m^2}{sin^2\frac{\phi}{2}}, \text{ or } 4E_1(E - E_1) = \frac{m^2}{sin^2\frac{\phi}{2}}$$

$$\text{from which we obtain } E_1^2 - EE_1 + \frac{m^2}{4\ sin^2\frac{\phi}{2}} = 0 \ .$$

[3]

Completing the square we find

$$\left(E_1 - \frac{E}{2}\right)^2 - \frac{E^2}{4} + \frac{m^2}{4 sin^2\frac{\phi}{2}} = 0 \text{ whence } E_1 = \frac{E}{2} \pm \frac{1}{2}\sqrt{E^2 - \frac{m^2}{sin^2\frac{\phi}{2}}} \ .$$

[3]

From the square root term we find $sin\frac{\phi}{2} = \frac{m}{E}$. **[2]**

4. Using the factor theorem three times

$$x^3 + 2x^2 - x - 2 = (x - 1)(x + 2)(x + 1) \ .$$

(Some students will do this in part by long division - full credit if they get the correct result.) **[2]**

$$\text{For } x^3 - x^2 - x - 2 \text{ the factor theorem gives } (x - 2) \text{ as a factor .}$$

Long division or other methods, which the students should do, leads to **[2]**

$$x^3 - x^2 - x - 2 = (x - 2)(x^2 + x + 1) \ .$$

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x + 1)(x - 1)(x^2 + 1) \ .$$

[2]

$$x^{\frac{3}{2}} - a^{\frac{3}{2}} = (x^{\frac{1}{2}})^3 - (a^{\frac{1}{2}})^3 = (\sqrt{x} - \sqrt{a})(x + \sqrt{x}\sqrt{a} + a) \ .$$

(The expansion for $x^3 \pm a^3$ has been done in the lectures.) **[4]**

5. We have

$$u = v_1 \cos\theta + \sqrt{2}v_2 \text{ and } v_1 \sin\theta = \sqrt{2}v_2 \ .$$

From the above

$$v_1 = \frac{u}{\sin\theta + \cos\theta} \quad \text{and} \quad v_2 = \frac{u \sin\theta}{\sqrt{2}(\sin\theta + \cos\theta)} \ .$$

Putting these into the third equation yields

$$u^2 = \frac{u^2}{(\sin\theta + \cos\theta)^2} + \frac{2u^2 \sin^2\theta}{2(\sin\theta + \cos\theta)^2}$$

i.e. $(\sin\theta + \cos\theta)^2 = (1 + \sin^2\theta)$, $\quad \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1 + \sin^2\theta$,

i.e. $2\sin\theta\cos\theta = \sin^2\theta$, $\quad \sin^2\theta(1-2\cot\theta) = 0$ and $\quad \sin\theta = 0$ or $\tan\theta = 2$.

The value $\tan\theta = 2$ corresponds to $\sin\theta = \frac{2}{\sqrt{5}}$ and $\theta = 63.4°$. **[6]**