PHAS1245: Mathematical Methods I - Problem Class 6

Week starting Monday 10th December

1. Use the standard expression for Maclaurin series to find the first few terms of the $1/(1+x^2)$ expansion. Then use the Maclaurin series

$$\frac{1}{1+x} = \sum_0^\infty (-x)^n$$

obtained in the lectures, to find the series for $1/(1+x^2)$ by replacing x above with x^2. and convince yourself that the two approaches are equivalent. Finally, use the above results to calculate the Maclaurin expansion for $f(x) = \arctan x$

2. Evaluate the following limits

$$\lim_{x\to 0} \frac{x\mathrm{e}^{-\mathrm{x}}}{1-\mathrm{e}^{-\mathrm{x}}} \qquad\qquad \lim_{x\to 0} \frac{\tan x - x}{\cos x - 1}\,.$$

3. Extend the technique used in the lectures for deriving the Taylor expansion of a single variable function, to a function of two variables $f(x, y)$ and show that the first and second order terms of the Taylor expansion around (x_0, y_0) are

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{1}{2!}\left[f_{xx}(\Delta x)^2 + 2f_{xy}\Delta x\Delta y + f_{yy}(\Delta y)^2\right]\,,$$

where

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial^2 f}{\partial x \partial y} \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2}\,,$$

$\Delta x = x - x_0$, $\Delta y = y - y_0$ and all the derivatives are calculated at (x_0, y_0). Then show that the terms in the square brackets can be rearranged to

$$\left[f_{xx}\left(\Delta x + \frac{f_{xy}\Delta y}{f_{xx}}\right)^2 + (\Delta y)^2\left(f_{yy} - \frac{(f_{xy})^2}{f_{xx}}\right)\right]$$

and determine the conditions for (x_0, y_0) to be a minimum.

4. A Fabry-Pérot interferometer consists of two parallel heavily silvered glass plates; light enters normally to the plates, and undergoes repeated reflections between them, with a small transmitted fraction emerging at each reflection. Find the intensity $|B|^2$ of the emerging wave, where

$$B = A(1-R)\sum_{n=0}^{\infty} R^n \mathrm{e}^{\imath n\phi}\,,$$

with A, R (the reflection coefficient, $0 < R < 1$) and ϕ real constants.

[Answer: $|B|^2 = A^2(1-R)^2/(1+R^2-2R\cos\phi)$]

5. The mean energy $\overline{E}$ of a quantum oscillator in thermal equilibrium is given by

$$\overline{E} = \frac{\sum_{n=0}^{\infty} kT\ nx\ \mathrm{e}^{-\mathrm{nx}}}{\sum_{n=0}^{\infty} \mathrm{e}^{-\mathrm{nx}}}$$

where $x = (hf)/(kT)$. Show that

$$\overline{E} = kT \frac{x\mathrm{e}^{-\mathrm{x}}}{1 - \mathrm{e}^{-\mathrm{x}}}\,.$$

What is the limiting value of $\overline{E}$ at small values of x? What is the limiting value of $\overline{E}$ at large values of x?