PHAS1245 - Problem Class 4 - Solutions

1.
$$\left(\frac{\partial\theta}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial\theta}\right)_y}$$

Since $\theta = \arctan(y/x) \Rightarrow \tan\theta = y/x \Rightarrow x = y/\tan\theta$, we have

$$\left(\frac{\partial x}{\partial\theta}\right)_y = \frac{-y}{\tan^2\theta}(\tan\theta)' = \frac{-y}{\tan^2\theta}\left(\frac{1}{\cos^2\theta}\right) = \frac{-y}{\tan^2\theta}\left(\tan^2\theta + 1\right) = -\frac{x^2+y^2}{y}\,.$$

Hence
$$\left(\frac{\partial\theta}{\partial x}\right)_y = -\frac{y}{x^2+y^2}\,.$$

Similarly we find that
$$\left(\frac{\partial\theta}{\partial y}\right)_x = \frac{x}{x^2+y^2}\,.$$

2. We have $z^2 = (x+\imath y)^2 = x^2 - y^2 + 2xy\imath$. So, $\mathrm{Re}(z^2) = x^2 - y^2$ and $\mathrm{Im}(z^2) = 2xy$. Hence the equation of the points in the complex plane satisfying $\mathrm{Re}(z^2) = \mathrm{Im}(z^2)$ is $x^2 - y^2 - 2xy = 0$.

3. We have
$$\frac{\partial f}{\partial x} = 3x^2 - 2y = 0 \Rightarrow y = \frac{3}{2}x^2$$
$$\frac{\partial f}{\partial y} = 3y^2 - 2x = 0 \Rightarrow \frac{3^3}{2^2}x^4 - 2x = 0 \Rightarrow 2x\left(\frac{3^3x^3}{2^3} - 1\right) = 0$$

Hence the points are $(0,0)$ and $(\frac{2}{3},\frac{2}{3})$. To determine their nature we need the second derivatives

$$\frac{\partial^2 f}{\partial x^2} = 6x \qquad \frac{\partial^2 f}{\partial y^2} = 6y \qquad \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 = (-2)^2 = 4$$

So, the point $(0,0)$ is undetermined since the two second derivatives are zero. The point $(\frac{2}{3},\frac{2}{3})$ is a minimum since the values of the two second derivatives are both positive $(4 > 0)$ and their product is 16,which is greater than 4, the value of the mixed derivative squared.

NOTE to the demonstrators: I think I did not make it clear in the lectures that when the second derivatives wrt x or y are zero, then the nature of the point cannot be determined with just the second derivatives (like in the case of single variable functions), so students may call the first point a saddle point. Please clearify this in the problem class if you get the chance.

4. We have
$$\mathrm{e}^{\imath\pi/12} = \mathrm{e}^{\imath(\pi/3-\pi/4)} = \mathrm{e}^{\imath\pi/3}\mathrm{e}^{-\imath\pi/4}$$
$$\Rightarrow \cos\frac{\pi}{12} + \imath\sin\frac{\pi}{12} = (\cos\frac{\pi}{3} + \imath\sin\frac{\pi}{3})(\cos\frac{\pi}{4} - \imath\sin\frac{\pi}{4})$$

$$\cos\frac{\pi}{12} + \imath\sin\frac{\pi}{12} = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} + \imath(\sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}).$$

Hence

$$\cot\frac{\pi}{12} = \frac{\cos\frac{\pi}{12}}{\sin\frac{\pi}{12}} = \frac{(1/2)(\sqrt{2}/2) + (\sqrt{3}/2)(\sqrt{2}/2)}{(\sqrt{3}/2)(\sqrt{2}/2) - (1/2)(\sqrt{2}/2)} = \frac{1+\sqrt{3}}{\sqrt{3}-1} = 2+\sqrt{3}.$$

5. We need to show that

$$\left(\frac{\partial f}{\partial v}\right)_u = 0.$$

The first thing one might think of trying is to express the above partial derivative using the "change of variables" rule:

$$\left(\frac{\partial f}{\partial v}\right)_u = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u.$$

To proceed like this, you would need to express x and y in terms of u and v and the algebra gets messy. Seeing that there is an equation connecting the partial derivatives wrt x and y, it is suggestive to try writing those out. We get

$$\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial f}{\partial u}\right)_v \left(\frac{\partial u}{\partial x}\right)_v + \left(\frac{\partial f}{\partial v}\right)_u \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial f}{\partial u}\right)_v 2x + \left(\frac{\partial f}{\partial v}\right)_u 2y,$$

$$\left(\frac{\partial f}{\partial y}\right)_x = \left(\frac{\partial f}{\partial u}\right)_v \left(\frac{\partial u}{\partial y}\right)_x + \left(\frac{\partial f}{\partial v}\right)_u \left(\frac{\partial v}{\partial y}\right)_x = \left(\frac{\partial f}{\partial u}\right)_v (-2y) + \left(\frac{\partial f}{\partial v}\right)_u 2x.$$

Now, multiplying the first of the above by y and the second by x and adding them together we get

$$0 = 2\left(\frac{\partial f}{\partial v}\right)_u (x^2+y^2).$$

Since the above has to be true for all values of (x, y), the derivative has to be equal to 0, q.e.d.

6. First write $-8+\imath 8\sqrt{3}$ in polar form: $r = \sqrt{8^2+3(8)^2} = 16$. Hence $(-8+\imath 8\sqrt{3}) = 16(-1/2+\imath)\sqrt{3}/2$. The angle that has $\cos\theta = -1/2$ and $\sin\theta = \sqrt{3}/2$ is $2\pi/3$ or $2\pi/3+2n\pi$, when n is any integer. Therefore

$$\sqrt[4]{-8+\imath 8\sqrt{3}} = \left(16\mathrm{e}^{\imath(2\pi/3+2n\pi)}\right)^{1/4} = 2\mathrm{e}^{\imath(2\pi/3+2n\pi)/4},$$

and the four roots are

$$n=0 \quad\Rightarrow\quad z_1 = \mathrm{e}^{\imath\pi/6},$$

$$n=1 \quad\Rightarrow\quad z_2 = \mathrm{e}^{\imath 4\pi/6},$$

$$n=2 \quad\Rightarrow\quad z_3 = \mathrm{e}^{\imath 7\pi/6},$$

$$n=3 \quad\Rightarrow\quad z_4 = \mathrm{e}^{\imath 10\pi/6}.$$