4 Integration

4.1 The definite and indefinite integrals

Definition of definite integral

The expression:

$$I_{ab} = \int_a^b f(x)dx = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x_i)\delta x \ .$$

is the definition of the definite integral of $f(x)$ between the lower limit $x = a$ and the upper limit $x = b$. It corresponds to the area under the curve.

Definition of indefinite integral

The indefinite integral is defined by

$$I(x) = \int_a^x f(u)du$$

where a is an arbitrary value of x and u is a dummy variable.

We now show that if we differentiate the indefinite integral just defined we get back to $f(x)$ ie integration is the reverse of differentiation and vice versa.
Using the basic definition of the derivative we find

$$\frac{dI(x)}{dx} = \frac{d}{dx}\int_a^x f(u)du = \lim_{\delta x \to 0} \frac{\int_a^{x+\delta x} f(u)du - \int_a^x f(u)du}{\delta x} = f(x)$$

(Think of the two integrals in the numerator in terms of areas $\sum f(x_i)\delta x$ where the first has one bin more than the second).
From now on we can regard the indefinite integral as the reverse of the derivative and we will write it as

$$\int f(x)dx$$

and not worry about limits a and x or the dummy variable u.
Of course, if F(x) is a function whose derivative is f(x), then all functions $F(x) + c$, where c is a constant, also have a derivative of $f(x)$. So we conventionally write

$$\int f(x)dx = F(x) + c$$

where c is the constant of integration.
Returning to the definite integral between two limits $x = A$ and $x = b$, this can be written as follows

$$\int_a^b f(x)dx = \int_{x0}^b f(x)dx - \int_{x0}^a f(x)dx = F(b) - F(a) \equiv [F]_a^b$$

where x_0 is an arbitrary fixed point.

4.2 Integrals of basic functions

In this section we obtain integrals by regarding integration as the reverse of differentiation. This method may be extended slightly by differentiating the result of an integration, and adjusting the result, so as to obtain the integrand that we started with.

The integral of x^n

Since we have

$$\frac{d\ x^n}{dx} = nx^{n-1} \quad \text{then} \quad \int \frac{d\ x^n}{dx} dx = x^n = \int nx^{(n-1)} dx \quad \text{or} \quad x^{n+1} = \int (n+1) x^n dx \quad \text{and}$$

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + constant$$

The integral of $1/x$

Although the curve of $1/x$ has clearly an area between it and the x axis the formula above fails for $n = -1$. We have seen that the function $\ln x$ has derivative given by

$$\frac{d \ln x}{dx} = \frac{1}{x}$$

thus the integral of $1/x$ is:

$$\int \frac{1}{x} dx = \ln\ |x| + c$$

Exponential functions

Considerations of differentiation have led to the identification

$$\frac{de^x}{dx} = e^x$$

therefore

$$\int e^x dx = e^x + c\ .$$

Consider now the integral of a^x. We first calculate its derivative. We notice that

$$\text{If} \quad y = a^x \quad \text{then} \quad \ln y = x \ln a$$

and

$$\frac{d \ln y}{dx} = \frac{d \ln y}{dy} \frac{dy}{dx} = \ln a.$$

Thus

$$\frac{1}{y} \frac{dy}{dx} = \ln a$$

and multiplying both sides by y

$$\frac{dy}{dx} = y \ln a\ .$$

Finally, replacing $y = a^x$ we obtain the wanted derivative

$$\boxed{\frac{da^x}{dx} = a^x \ln a}$$

Since

$$\frac{d\ a^x}{dx} = a^x ln\ a, \int \frac{d\ a^x}{dx} dx = \int a^x \ln a dx$$

and

$$\boxed{\int a^x dx = \frac{a^x}{\ln a} + c.}$$

<u>Trigonometric functions</u>

We have seen that:

$$\frac{d \sin\theta}{d\theta} = \cos\theta\ .$$

$$\frac{d \cos\theta}{d\theta} = -\sin\theta\ .$$

From the above we also have

$$\int \sin\theta d\theta = -\cos\theta + c$$

and

$$\int \cos\theta d\theta = \sin\theta + c\ .$$

By taking derivatives it is immediate to verify that:

$$\int \tan\theta d\theta = -\ln|\cos\theta| + c$$

$$\int \cos\theta \sin^n\theta d\theta = \frac{\sin^{n+1}\theta}{n+1} + c$$

$$\int \sin\theta \cos^n\theta d\theta = \frac{\cos^{n+1}\theta}{n+1} + c$$

<u>Integrals producing inverse trigonometric functions</u>

We have seen that:

$$\frac{d \sin^{-1}(x/a)}{dx} = \frac{1}{\sqrt{a^2 - x^2}}\ ;$$

$$\frac{d \cos^{-1}(x/a)}{dx} = \frac{-1}{\sqrt{a^2 - x^2}}\ ;$$

$$\frac{d\tan^{-1}(x/a)}{dx} = \frac{a}{a^2+x^2} \; .$$

Thus it is immediate to verify that:

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}(x/a) + c$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}dx = \cos^{-1}(x/a) + c$$

$$\int \frac{a}{a^2+x^2}dx = \tan^{-1}(x/a) + c$$

Integration of some hyperbolic functions.

We have seen that:

$$\frac{d\sinh x}{dx} = \cosh x;$$

$$\frac{d\cosh x}{dx} = \sinh x;$$

from which it follows that:

$$\int \sinh x dx = \cosh x;$$

$$\int \cosh x dx = \sinh x;$$

$$\int \tanh x dx = \ln|\cosh x| \; .$$

Integrals producing inverse hyperbolic functions

We saw that:

$$\frac{d\sinh^{-1}(x/a)}{dx} = \frac{1}{\sqrt{x^2+a^2}} \; .$$

The above can be integrated, thus giving:

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a}$$

Likewise from

$$\frac{d}{dx}\tanh^{-1}\frac{x}{a} = \frac{a}{a^2-x^2}$$

it follows that

$$\int \frac{a\ dx}{a^2 - x^2} = \tanh^{-1} \frac{x}{a}$$

<u>Example I</u>

$$I = \int \sin^5 x dx$$

We need to reduce it to several of the above forms.

$$\begin{aligned}
\int \sin^5 x dx &= \int \sin^4 x \sin x dx && (1)\\
&= \int (1 - \cos^2 x)^2 \sin x dx && (2)\\
&= \int (1 - 2\cos^2 x + \cos^4 x) \sin x dx && (3)\\
&= \int (\sin x - 2\cos^2 x \sin x + \cos^4 x \sin x) dx && (4)\\
&= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c\ . && (5)
\end{aligned}$$

<u>Example II</u>

$$I = \int_{-\pi}^{\pi} x \cos x dx$$

We recall the definition of odd and even functions.

An odd function satisfies the condition $f(x) = -f(-x)$.

An even function satisfies the condition $f(x) = f(-x)$.

The definite integral of an odd function from $x = -a$ to $x = +a$ is zero because the area calculated for x less than zero is equal but of opposite sign to the area for x greater than zero.

Thus

$$\int_{-\pi}^{\pi} x \cos x dx = 0\ .$$

<u>Example III</u>

$$I = \int \frac{1}{3x + 2} dx$$

Substitute

$$u = 3x + 2 \Rightarrow du = 3dx$$

We can now determine the integral

$$I = \int \frac{1}{u}\frac{du}{3} = \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln|3x + 2| + c$$

Example IV

$$I = \int \sin x \cos x dx$$

We could use the standard formulae for $\int \sin x \cos^n x dx$. However, it can also be easily solved directly. By substituting

$$u = \sin x \Rightarrow du = \cos x dx$$

we obtain

$$I = \int u du = \frac{u^2}{2} + c = \frac{1}{2}\sin^2 x + c$$

Example V

$$I = \int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

By substituting

$$u = \cos x \Rightarrow du = -\sin x dx$$

we obtain

$$I = \int \frac{-1}{u} du = -\ln|u| + c = -\ln|\cos x| + c \ .$$

4.3 Integration using partial fractions

Reminder: partial fractions

First let us consider expressing proper fractional functions in terms of partial fractions [1] (We shall return to improper fractional functions shortly). The procedure we adopt is justified by considering that denominators like $(x+1)^2$ could result from partial fractions with a denominator of the form $(x+1)$ as well as its square. Also quadratic denominators will in general have terms linear in x in the numerator. We do a series of illustrative examples.

Example 1.

$$\frac{x+3}{(x-2)(x+4)} \equiv \frac{A}{(x-2)} + \frac{B}{(x+4)} \equiv \frac{A(x+4)+B(x-2)}{(x-2)(x+4)} \text{ ie } x+3 = A(x+4)+B(x-2) \ .$$

Choosing $x = 2$ to eliminate the B term we find $A = \frac{5}{6}$.
Similarly choosing $x = -4$ to eliminate A, we find $B = \frac{1}{6}$.

Example 2.

$$\frac{x^2-3}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \equiv \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)} \quad \text{and}$$

$$x^2 - 3 \equiv A(x^2+1) + (Bx+C)(x-1)$$

[1] In a 'proper' fraction the degree of the numerator is less than that of the denominator. For an 'improper' fraction the degree of the numerator is greater or equal to that of the denominator.

Substituting $x = 1$ so as to eliminate B and C gives $A = -1$ and substituting $x = 0$ will eliminate B. Since $A = -1$ we find $C = 2$ and substituting any other value for x, a small value is sensible, will determine $B = 2$.

Example 3.

$$\frac{x-1}{(x+1)(x-2)^2} \equiv \frac{A}{x+1} + \frac{B}{x-2} + \frac{D}{(x-2)^2}$$

Choosing the obvious values for x determines D to be $\frac{1}{3}$ and A to be $-\frac{2}{9}$. Then comparing the coefficients of x^2 determines B to be $\frac{2}{9}$.

Example 4.

Since $\dfrac{x^3}{(x+1)(x-3)}$ is an improper fraction we must do a long division first.

The result of dividing $x^2 - 2x - 3$ into x^3 is $x + 2$ with a remainder we need not determine, and we have

$$\frac{x^3}{(x+1)(x-3)} \equiv x + 2 + \frac{\mathit{Remainder}}{(x+1)(x-3)} \equiv x + 2 + \frac{A}{x+1} + \frac{B}{x-3}$$

Obvious substitutions determine $A = \frac{1}{4}$ and $B = \frac{27}{4}$.

We now consider the use of partial franctions for integration. Integration sometimes requires fractional functions to be expressed in terms of two or more simpler fractional functions. If we can split up complicated fractions into several simpler forms, we can then do the integration more easily.

Best demonstrated with an example:

$$\begin{aligned} I &= \int \frac{x+3}{(x-2)(x+4)} = && (6) \\ &= \int \frac{5}{6(x-2)}dx + \int \frac{1}{6(x+4)}dx && (7) \\ &= \frac{5}{6}\ln|x-2| + \frac{1}{6}\ln|x+4| + c && (8) \end{aligned}$$

When considering a quadratic polynom as the denominator, i.e.

$$I = \int \frac{dx}{ax^2 + bx + c} \ .$$

The way we solve this depends on the discriminant $b^2 - 4ac$.

(a) If $b^2 - 4ac > 0$, we can decompose the original fraction into partial fractions.

(b) If $b^2 - 4ac < 0$, then we will need to resort to trigonometric substitutions.

c) If $b^2 - 4ac = 0$ then we have:

$$ax^2 + bx + c = a(x + \frac{b}{2a})^2$$

and

$$\int \frac{dx}{ax^2 + bx + c} = \int \frac{dx}{a(x + b/2a)^2} = -\frac{1}{a}\frac{1}{(x + \frac{b}{2a})} + c$$

Example

$$I = \int \frac{dx}{(x-1)^2 + 4}$$

We substitute $u = (x-1)$ thus $du = dx$, and the integral becomes

$$I = \int \frac{du}{u^2 + 4}$$

We recall the standard integral

$$\int \frac{a}{a^2 + x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$$

from which we can derive:

$$I = \frac{1}{2}\tan^{-1}\frac{u}{2} + c = \frac{1}{2}\tan^{-1}\frac{x-1}{2} + c$$

4.4 Integration by parts

Integration by parts is the analogue of the product rule for differentiation and indeed we start from the product rule

$$(uv)' = u'v + uv'$$

Now integrate:

$$\int (uv)' dx = \int u'v dx + \int uv' dx$$

But

$$\int (uv)' dx = uv$$

Thus rearranging

$$\boxed{\int uv' dx = uv - \int u'v dx}$$

Therefore for a product we assign the first term as u and the second as v' where the former should be easy to differentiate and the latter easy to integrate. And the results $u'v$ should be easier to integrate.

Example I

$$I = \int xe^x dx$$

We identify u and v as follows:

$$u = x \Rightarrow u' = 1 \tag{9}$$
$$v = e^x \Rightarrow v' = e^x \tag{10}$$

Thus
$$I = xe^x - \int 1 \times e^x dx = (x-1)e^x$$

<u>Example II</u>

$$I = \int \tan^{-1} x dx$$

We identify u and v as follows:

$$u = \tan^{-1} x \Rightarrow u' = \frac{1}{1+x^2} \quad (11)$$
$$v = x \Rightarrow v' = 1 \quad (12)$$

Thus
$$I = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$$

To determine the second integral, we set
$$t = 1 + x^2 \Rightarrow dt = 2x dx$$

$$I_2 \equiv \int \frac{x}{1+x^2} dx = \int \frac{1}{t} \frac{dt}{2} = \frac{1}{2} \ln t = \frac{1}{2} \ln(1+x^2)$$

And we have
$$I = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + c \ .$$

<u>Example III</u>
Consider
$$\int \ln x dx$$

Now we set
$$\begin{aligned} u &= \ln x \Rightarrow du = \frac{dx}{x} \\ v &= x \Rightarrow v' = 1 \end{aligned} \quad (13)$$

$$\text{Now} \quad \int \ln x \ dx = x \ln x - \int x \frac{dx}{x} = x \ln x - \int dx = x(\ln x - 1) \ .$$

4.5 Integration using differentiation with respect to a parameter

Example:
$$I = \int xe^{-ax} dx$$

We notice that
$$xe^{-ax} = -\frac{d}{da} e^{-ax}$$

hence

$$I = -\int \frac{d}{da} e^{-ax} dx \ .$$

As the differentiation with respect to a is an independent operation from the integration with respect to x, we can do whichever we prefer first and then the other. So

$$\begin{aligned}
I &= -\frac{d}{da}\int e^{-ax}dx = & (14)\\
&= -\frac{d}{da}\left(\frac{e^{-ax}}{-a}\right) = & (15)\\
&= \frac{d}{da}\left(\frac{e^{-ax}}{a}\right) = & (16)\\
&= -\frac{xe^{-ax}}{a} - \frac{e^{-ax}}{a^2} = & (17)\\
&= -e^{-ax}\frac{1+ax}{a^2} & (18)
\end{aligned}$$

4.6 Integration using reduction formulae

By being able to evaluate a simple expression, e.g.

$$\int_0^1 (1-x^3)^0 dx$$

we can use reduction formulae to evaluate a more complicated integral, e.g.

$$\int_0^1 (1-x^3)^4 dx$$

By using integration by parts we want to find a relationship between I_n and I_{n-1} where

$$I_n = \int_0^1 (1-x^3)^n dx$$

and n is a positive integer.

$$\begin{aligned}
I_n &= \int_0^1 (1-x^3)(1-x^3)^{n-1}dx & (19)\\
&= \int_0^1 (1-x^3)^{n-1}dx - \int_0^1 x^3(1-x^3)^{n-1}dx & (20)\\
&= I_{n-1} - \int_0^1 xx^2(1-x^3)^{n-1}dx & (21)
\end{aligned}$$

We can evaluate the 2nd term by integrating by parts. We set

$$\begin{aligned}
u &= x \Rightarrow u' = 1 & (22)\\
v &= \frac{(1-x^3)^n}{-3n} \Rightarrow v' = x^2(1-x^3)^{n-1} & (23)
\end{aligned}$$

and obtain

$$I_n = I_{n-1} + \left[\frac{x}{3n}(1-x^3)^n\right]_0^1 - \int_0^1 \frac{1}{3n}(1-x^3)^n dx$$

$$I_n = I_{n-1} - \frac{1}{3n}I_n \Rightarrow I_n = \frac{3n}{3n+1}I_{n-1}$$

Evaluating I_0 is immediate:

$$I_0 = \int_0^1 (1-x^3)^0 dx = \int_0^1 dx = [x]_0^1 = 1$$

from which all the others follow:

$$I_1 = \frac{3\times 1}{3\times 1+1} \times 1 = \frac{3}{4}$$

$$I_2 = \frac{6}{7}I_1, \qquad I_3 = \frac{9}{10}I_2, \qquad I_4 = \frac{12}{13}I_3$$

$$I_2 = \frac{12}{13} \times \frac{9}{10}\frac{6}{7}\frac{3}{4} = \frac{1944}{3640} = \frac{243}{455}$$

4.7 Average of a distribution [see Riley et al, 2.2.13]

Average of a function

The average of y_1, y_2, y_3, ..., y_n is

$$\bar{y} = \frac{1}{n}\sum_{k=1}^{n} y_k$$

For a function f we sample values at equally spaced points in $[a,b]$: $x_1, x_2, ...x_n$. The spacing is $\Delta x = (b-a)/n$. Thus we have

$$\bar{f} = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$$

We notice that

$$\frac{1}{n} = \frac{\Delta x}{b-a}$$

thus $\bar{f}$ can be written as

$$\bar{f} = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\Delta x$$

And for $n \to \infty$,

$$\boxed{\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx}$$

Example: determine the time-averaged electrical power generated by a current $I = I_0\cos(2\pi f t)$ and a voltage $V = V_0\cos(2\pi f t + \delta)$, which is phase shifted, over one cycle, i.e. from $t=0$ to $t = T = 1/f$.

$$
\begin{aligned}
P &= \langle IV \rangle = \frac{1}{T}\int_0^T I_0 V_0 \cos(2\pi f t)\cos(2\pi f t+\delta) \\
&= \frac{I_0V_0}{T}\int_0^T \cos(2\pi f t)[\cos 2\pi f t\cos\delta - \sin(2\pi f t)\sin\delta]dt \\
&= \frac{I_0V_0}{T}\int_0^T \cos^2(2\pi f t)\cos\delta - \cos(2\pi f t)\sin(2\pi f t)\sin\delta]dt \\
&= \frac{I_0V_0}{T}\int_0^T [\frac{1+\cos(4\pi f t)}{2}\cos\delta - \frac{\sin(4\pi f t)}{2}\sin\delta]dt \\
&= \frac{I_0V_0}{T}\left\{\left[\frac{\cos\delta}{2}t\right]_0^T + \left[\frac{\sin(4\pi f t)}{2\times 4\pi f}\cos\delta\right]_0^T + \left[\frac{\cos(4\pi f t)}{2\times 4\pi f}\sin\delta\right]_0^T\right\} = \qquad (24) \\
&= \frac{I_0V_0}{2}\cos\delta \qquad (25)
\end{aligned}
$$

<u>Average for a discrete variable probability distribution</u>
Let x be a discrete random variable taking the values x_1,..., x_n with probability $P(x_1)$, $P(x_2)$, ..., $P(x_n)$. The mean, or expectation or expected value of x, is defined as

$$\boxed{\bar{x} = \sum_{i=1}^{n} x_i P(x_i)}\;.$$

Example: if x can take the values 1,2,3,4 and 5 with probabilities $P(1) = 1/10$, $P(2) = 2/10$, $P(3) = 4/10$, $P(4) = 2/10$, $P(5) = 1/10$, then

$$\bar{x} = \frac{1}{10} + \frac{4}{10} + \frac{12}{10} + \frac{8}{10} + \frac{5}{10} = 3$$

<u>Average for a continuous variable probability distribution</u>

We can use integration to find the mean (or expectation) value of a variable in a probability function. To find the mean value of x in a function between the limits a and b, we do the following integration

$$\boxed{\bar{x} = \int_a^b x f(x) dx}$$

Example: take the Maxwell-Boltzmann distribution, which describes the distribution of the magnitude of speed for non-interacting particles in a gas:

$$P(v) = Av^2 e^{-B^2 v^2}$$

where

$$A = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}, \qquad B^2 = \frac{m}{2kT}$$

From which we can derive the average of v:

$$\bar{v} = \int_0^{+\infty} v \times A v^2 e^{-B^2 v^2} dv$$

We can integrate by parts:

$$\begin{aligned} u &= v^2 \Rightarrow u' = 2v && (26) \\ w &= -\frac{e^{-B^2 v^2}}{2B^2} \Rightarrow w' = v e^{-B^2 v^2} && (27) \end{aligned}$$

$$\begin{aligned} \bar{v} &= A\left[-v^2 \frac{e^{-B^2 v^2}}{2B^2}\right]_0^{+\infty} + \frac{A}{B^2}\int_0^{+\infty} v e^{-B^2 v^2} dv && (28) \\ &= 0 + \frac{A}{B^2}\left[\frac{e^{-B^2 v^2}}{-2B^2}\right]_0^{+\infty} && (29) \\ &= \frac{A}{2B^4} && (30) \end{aligned}$$

Replacing with values of A and B we obtain

$$\bar{v} = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \times \frac{1}{2} \times \left(\frac{2kT}{m}\right)^2 = \sqrt{\frac{8kT}{\pi m}}$$

4.8 Volumes of revolution [see Riley et al, 2.2.13]

One can use integration to find the volume of a solid generate when a curve $f(x)$ is rotated about the $x-$axis. We can think of the volume as being composed of disks with radius $f(x)$ and thickness Δx, and hence the volume

$$V = \pi f(x)^2 \Delta x$$

for one disk. To find the total volume, we sum up all disks between the limits $x = 0$ and $x = b$. So the volume of revolution is

$$V = \sum_{x=a}^{x=b} \pi f(x)^2 \Delta x$$

which becomes the following equality when $\Delta x \to 0$

$$V = \int_a^b \pi f(x)^2 dx$$

Example: derive the volume of a sphere of radius r.

A sphere is formed by rotating a disk around the x axis, which has the equation

$$x^2 + y^2 = r^2, \qquad \text{i.e.} \qquad y^2 = r^2 - x^2 \ .$$

Putting this into the volume of revolution formula, we have:

$$\begin{aligned} V_{\text{sphere}} &= \pi \int_{-r}^{+r} (r^2 - x^2)dx && (31) \\ &= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{+r} && (32) \\ &= \pi \left(r^3 - \frac{r^3}{3} \right) - \pi \left(-r^3 + \frac{r^3}{3} \right) = \frac{4}{3}\pi r^3 && (33) \end{aligned}$$

4.9 Numerical integration and the trapezium rule [see Riley et al, 27.4.1]

We divide the area under a curve which has a range $[a, b]$ into N strips of equal width.

The thickness, h, is defined by

$$h = \frac{b-a}{N}$$

As each strip, spanning y_{i-1} to y_i is a trapezium and we know its area we can write

$$\Delta A = \frac{1}{2}(y_{i-1} + y_i)\frac{b-a}{N}$$

and summing up all trapezia

$$A = \int_a^b f(x)dx = \frac{b-a}{N}\left[\frac{1}{2}y_0 + (y_1 + y_2 + ... + y_{N-1} + \frac{1}{2}y_N\right]$$

4.10 Applications of complex numbers to integration [see Riley 3.6]

We can use complex numbers to perform differentiation or integration which should make the process simpler. For example, consider the integral

$$I = \int e^{ax} \cos bx dx \ .$$

We could do integration by parts twice. Or, note that

$$e^{ibx} = \cos bx + i \sin bx$$

thus

$$e^{ax}(\cos bx + i \sin bx) = e^{ax}e^{ibx} = e^{(a+ib)x}$$

where the real part is the integrand we want.

$$\begin{aligned} \int e^{(a+ib)x} &= = \frac{e^{(a+ib)x}}{(a+ib)} + constant = \\ &= \frac{e^{(a+ib)x}}{(a+ib)}\frac{a-ib}{(a-ib)} + constant = \frac{e^{ax}}{(a^2+b^2)}(\cos bx + i \sin bx)(a-ib) + constant = \\ &= \frac{e^{ax}}{(a^2+b^2)}[a \cos bx + b \sin bx + i(a \sin bx - b \cos bx)] + constant && (34) \end{aligned}$$

Thus

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{(a^2+b^2)}[a\cos bx + b\sin bx] + constant$$

and we also get

$$\int e^{ax}\sin bx dx = \frac{e^{ax}}{(a^2+b^2)}[a\sin bx - b\cos bx] + constant$$

4.11 Various examples [see Riley et al, 2.2.7]

More trigonometric substitution

Integrals of the form

$$I = \int \frac{dx}{a+b\cos x}dx \qquad \text{and} \qquad I = \int \frac{dx}{a+b\sin x}dx$$

can be solved by making the substitution

$$t = \tan\frac{x}{2}\ .$$

From before,

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}(1+\tan^2\frac{x}{2}) = \frac{1+t^2}{2}$$

i.e.

$$dx = \frac{2dt}{1+t^2}\ .$$

And $\sin x$, $\cos x$ can be expressed in terms of t as

$$\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)} = \frac{2t}{1+t^2} \qquad (35)$$

$$\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-t^2}{1+t^2} \qquad (36)$$

Example

$$I = \int_{-\pi/2}^{\pi/2}\frac{2}{1+3\cos x}dx$$

First we do the indefinite integral.

$$t = \tan\frac{x}{2} \Rightarrow dx = \frac{2}{1+t^2}dt$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$
\begin{aligned}
I &= \int \frac{2}{1+3\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt \\
&= \int \frac{4}{1+t^2+3(1-t^2)} dt \\
&= \int \frac{2}{2-t^2} dt \\
&= \int \frac{2}{(\sqrt{2}-t)(\sqrt{2}+t)} dt \\
&= \frac{1}{\sqrt{2}} \int \left(\frac{1}{\sqrt{2}-t} + \frac{1}{\sqrt{2}+t} \right) dt \\
&= -\frac{1}{\sqrt{2}} \ln(\sqrt{2}-t) + \frac{1}{\sqrt{2}} \ln(\sqrt{2}+t) \\
&= \frac{1}{\sqrt{2}} \ln\left(\frac{\sqrt{2}+t}{\sqrt{2}-t} \right) \\
&= \frac{1}{\sqrt{2}} \ln\left(\frac{\sqrt{2}+\tan(x/2)}{\sqrt{2}-\tan(x/2)} \right)
\end{aligned}
$$

Now the definite integral

$$
\begin{aligned}
I &= \frac{1}{\sqrt{2}} \left[\ln\left(\frac{\sqrt{2}+\tan(x/2)}{\sqrt{2}-\tan(x/2)} \right) \right]_{-\pi/4}^{\pi/4} \\
&= \frac{1}{\sqrt{2}} \left(\ln\left(\frac{\sqrt{2}+\tan(\pi/4)}{\sqrt{2}-\tan(\pi/4)} \right) - \ln\left(\frac{\sqrt{2}+\tan(-\pi/4)}{\sqrt{2}-\tan(-\pi/4)} \right) \right) \qquad (37)
\end{aligned}
$$

Example

$$
I = \int \frac{1}{3+\cos^2 x} dx
$$

We change variable

$$
t = \tan x \Rightarrow \frac{dt}{dx} = 1 + \tan^2 x \Rightarrow dx = \frac{dt}{1+t^2}
$$

$$
\begin{aligned}
I &= \int \frac{1+t^2}{4+3t^2}\frac{dt}{1+t^2} \\
&= \int \frac{dt}{4+3t^2} \\
&= \frac{1}{3}\int \frac{dt}{4/3+t^2} \\
&= \frac{1}{3}\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{t}{\frac{2}{\sqrt{3}}}\right)+c \\
&= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{2}\tan x\right)+c \qquad (38)
\end{aligned}
$$