3 Differentiation

3.1 Definitions

Definition of limit

Consider the function $f(x)$. If we can make $f(x)$ as near as we want to a given number l by choosing x sufficiently near to a number a, then l is said to be the limit of $f(x)$ as $x \to a$ and it is written as

$$\lim_{x \to a} f(x) = l \ .$$

The Derivative

The derivative of $f(x)$ is the slope of, or the gradient of the tangent to, the function $f(x)$ at x and is given by

$$f'(x) \equiv \frac{df}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Similarly, the second derivative is

$$f''(x) \equiv \frac{d^2 f}{dx^2} = \lim_{\delta x \to 0} \frac{f'(x+\delta x) - f'(x)}{\delta x}$$

and generally

$$f^{(n)}(x) \equiv \frac{d^n f}{dx^n} = \lim_{\delta x \to 0} \frac{f^{(n-1)}(x+\delta x) - f^{(n-1)}(x)}{\delta x}$$

where $f'(x) \equiv f^{(1)}(x)$, etc and $f^{(0)}(x) = f(x)$.

3.2 Examples of derivations of derivatives

The Derivative of x^n

From the above definition

$$\frac{d\ x^n}{dx} = \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{\delta x} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

Now recall that

$$a^n - b^n = (a-b)(a^{(n-1)} + a^{(n-2)}b + a^{(n-3)}b^2 ......a\ b^{(n-2)} + b^{(n-1)})$$

Applying this last result with $a = x + h$ and $b = x$ we get

$$\frac{d\ x^n}{dx} = \lim_{h \to 0}\ h\ \frac{((x+h)^{(n-1)} + (x+h)^{n-2}\ x + (x+h)^{n-3}x^2 ....(x+h)x^{(n-2)} + x^{(n-1)})}{h}$$

Thus

$$\frac{d\ x^n}{dx} = nx^{(n-1)}$$

Note that this derivation can be extended to negative and fractional values of n - we shall assume without proof that the above result is valid for all values of n.

The derivative of $y = e^x$

Consider first the more general function $f(x) = a^x$. The definition of the derivative gives

$$\frac{d\ a^x}{dx} = \lim_{\delta x \to 0} \frac{(a^{(x+\delta x)} - a^x)}{\delta x} = \lim_{\delta x \to 0} a^x\ \frac{(a^{(\delta x)} - 1)}{\delta x}$$

In words this result states that the derivative of a^x equals a^x times the slope of a^x at $x = 0$ (this slope is the $\frac{(a^{(\delta x)}-1)}{\delta x}$ ie the $\frac{(a^{(0+\delta x)}-a^0)}{\delta x}$ term. That is:

$$f'(x) = a^x f'(0) \tag{1}$$

Note also that although we recognise this term as the slope, we do not get an analytic expression for this slope. So its natural to ask the question whether a value of a exists such that at $x = 0$ the slope is 1. The number for which $f'(0) = 1$ is given the name "e", after the mathematician Euler. Thus the function e^x is defined such that its slope at $x = 0$ equals unity and hence from the expression for the derivative of a^x we have for $f(x) = e^x$

$$y = \frac{dy}{dx} = e^x$$

ie e^x is a function which equals its slope.

Derivative of $\sin\theta$ and $\cos\theta$

If $f = \sin\theta$ then from the fundamental definition of the derivative

$$\frac{df}{d\theta} = \lim_{\delta\theta \to 0} \frac{\sin(\theta + \delta\theta) - \sin\theta}{\delta\theta}$$

$$= \lim_{\delta\theta \to 0} \frac{\sin\theta\cos\delta\theta + \cos\theta\sin\delta\theta - \sin\theta}{\delta\theta}$$

$$= \cos\theta\ .$$

$$\left(\cos\delta\theta \to 1, \quad \frac{\sin\delta\theta}{\delta\theta} \to 1 \text{ as } \delta\theta \to 0\right)$$

If $f = \cos\theta$ then

$$\frac{df}{d\theta} = \lim_{\delta\theta \to 0} \frac{\cos(\theta + \delta\theta) - \cos\theta}{\delta\theta}$$

$$= \lim_{\delta\theta \to 0} \frac{\cos\theta\cos\delta\theta - \sin\theta\sin\delta\theta - \cos\theta}{\delta\theta}$$

$$= -\sin\theta \ .$$

3.3 Derivatives of basic functions

Differentiation from first principles is time-consuming. What we usually do is use a list of derivatives of basic functions as a basis for more complicated functions. A set of useful derivatives is the following:

$$\frac{dx^n}{dx} = nx^{n-1} \tag{2}$$

$$\frac{de^{ax}}{dx} = ae^{ax} \tag{3}$$

$$\frac{d\ln(x)}{dx} = \frac{1}{x} \tag{4}$$

$$\frac{d\sin(ax)}{dx} = a\cos ax \tag{5}$$

$$\frac{d\cos(ax)}{dx} = -a\sin(ax) \tag{6}$$

$$\frac{d\sec(ax)}{dx} = a\sec(ax)\tan(ax) \qquad [\text{reminder} : \sec x \equiv \frac{1}{\cos x}] \tag{7}$$

$$\frac{d\tan(ax)}{dx} = a\sec^2(ax) \tag{8}$$

$$\frac{d\text{cosec}(ax)}{dx} = -a\text{cosec}(ax)\cot ax \qquad [\text{reminder} : \text{cosec}x \equiv \frac{1}{\sin x}] \tag{9}$$

$$\frac{d\cot(ax)}{dx} = -a\text{cosec}^2 ax \qquad [\text{reminder} : \cot x \equiv \frac{\cos x}{\sin x}] \tag{10}$$

$$\frac{d\sin^{-1}(x/a)}{dx} = \frac{1}{\sqrt{a^2 - x^2}} \tag{11}$$

$$\frac{d\cos^{-1}(x/a)}{dx} = \frac{-1}{\sqrt{a^2 - x^2}} \tag{12}$$

$$\frac{d\tan^{-1}(x/a)}{dx} = \frac{a}{a^2 + x^2} \tag{13}$$

3.4 The product rule and derivative of quotients

The product rule

For

$$f(x) = u(x)v(x)$$

what is $f'(x)$? We start from the definition

$$\frac{df}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Now

$$f(x+\delta x) - f(x) = u(x+\delta x)v(x+\delta x) - u(x)v(x) \tag{14}$$
$$= u(x+\delta x)\left[v(x+\delta x) - v(x)\right] + \left[u(x+\delta x) - u(x)\right]v(x) \tag{15}$$

and we have

$$\frac{df}{dx} = \lim_{\delta x \to 0} \{u(x+\delta x)\left[\frac{v(x+\delta x)-v(x)}{\delta x}\right] + \left[\frac{u(x+\delta x)-u(x)}{\delta x}\right]v(x)\}$$

In the limit $\delta x \to 0$ the factors in the square brackets become $v'(x)$ and $u'(x)$ and $u(x+\delta x)$ become $u(x)$. Thus we have

$$\boxed{\frac{df}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.} \tag{16}$$

or more compactly using a different notation

$$(uv)' = uv' + u'v \tag{17}$$

This rule can be extended to the product of three or more functions:

$$(uvw)' = u'vw + uv'w + uvw' \tag{18}$$

An example

$$\frac{d}{dx}\left(x^3 \sin x\right) = x^3\frac{d}{dx}(\sin x) + \frac{d}{dx}\left(x^3\right)\sin x$$

$$= x^3 \cos x + 3x^2 \sin x \ .$$

<u>The derivative of a quotient</u>

Applying the product rule to a quotient of two factors

$$f(x) = \frac{u(x)}{v(x)} \qquad \text{or} \qquad f(x) = u(x)\frac{1}{v(x)} \ .$$

So

$$f'(x) = (uv^{-1})' = u'v^{-1} + u(v^{-1})' = \tag{19}$$
$$= \frac{u}{v} + u\left(\frac{-v}{v^2}\right) = \tag{20}$$
$$= \frac{u'v - uv'}{v^2} \tag{21}$$

<u>Examples</u>

For $f = \tan\theta$ use $f = \frac{\sin\theta}{\cos\theta}$.

Then

$$\frac{df}{d\theta} = \frac{\cos\theta\cos\theta + \sin\theta\sin\theta}{\cos^2\theta}$$

$$= \frac{\sin^2\theta}{\cos^2\theta} + 1 = 1 + \tan^2\theta = \sec^2\theta \; .$$

If $f = \text{cosec}\theta = \frac{1}{\sin\theta}$ then

$$\frac{df}{d\theta} = \frac{(-1)\cos\theta}{(\sin\theta)^2} = -\csc\theta\cot\theta$$

Similarly if $f = \sec\theta$, then

$$\frac{df}{d\theta} = \sec\theta\tan\theta.$$

3.5 The chain rule

We may have a situation where we need to differentiate a function of a function, i.e. we have a situation where $f(x)$ can be expressed as $f = f(u(x))$. For example

$$f(x) = (3 + x^2)^3 = u(x)^3 \tag{22}$$

where

$$u(x) = 3 + x^2$$

To differentiate such functions we use the chain rule

$$\boxed{\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} \; .}$$

For our example

$$f(x) = \left(3 + x^2\right)^3 = f(u) \; , \;\; u = \left(3 + x^2\right)$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = 3u^2 \cdot 2x = 3\left(3 + x^2\right)^2 \times 2x$$

$$= 6x\left(3 + x^2\right)^2 \; .$$

Three more examples:

- $f(x) = e^{ax}$ [one of our standard functions from few pages ago]

$$f(u) = e^u \qquad \text{and} \qquad u = ax$$

$$\frac{df}{du} = e^u \qquad \text{and} \qquad \frac{du}{dx} = a \;\; \text{thus} \qquad \frac{df}{dx} = ae^{ax}$$

- $f(x) = (\sin x)^3$

  We have

$$f(u) = u^3 \qquad \text{and} \qquad u = \sin x$$

$$\frac{df}{du} = 3u^2 \qquad \text{and} \qquad \frac{du}{dx} = \cos x \quad \text{thus} \qquad \frac{df}{dx} = 3\sin^2 x \cos x$$

- $f(x) = \sin(x^3)$

  We have

$$f(u) = \sin(u) \qquad \text{and} \qquad u = x^3$$

$$\frac{d}{du} = \cos u \qquad \text{and} \qquad \frac{du}{dx} = 3x^2 \quad \text{thus} \qquad \frac{df}{dx} = 3x^2 \cos x^3$$

3.6 Implicit differentiation

So far we have only considered functions where $y = f(x)$, i.e. only one variable on the right-hand side. We may have a situation where it is not so easy to express y in terms of x, e.g.:

$$x^3 - 3xy + y^3 = 2 \ .$$

We therefore differentiate term by term with respect to x which is called implicit differentiation.

$$\frac{d}{dx}(x^3) - \frac{d}{dx}(3xy) + \frac{d}{dx}(y^3) = \frac{d}{dx}(2)$$

$$(3x^2) - \left[3x\frac{dy}{dx} + 3y\right] + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

3.7 Parametric differentiation and inverse differentiation

Inverse differentiation

If $y = f(x)$ and $x = f^{-1}(y)$ are inverse functions, then

$$\boxed{\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}} \tag{23}$$

Example: inverse trigonometric functions

Differentiate $y = \sin^{-1} x$. We see that $x = \sin y$.

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} \tag{24}$$

Now, we know $\sin^2 y + \cos^2 y = 1$, thus

$$\cos y = \sqrt{1 - \sin^2 y} \tag{25}$$

so

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}} \tag{26}$$

but $x = \sin y$ thus

$$\frac{d \sin^{-1} x}{dx} = \frac{1}{\sqrt{1 - x^2}} \tag{27}$$

Example: inverse hyperbolic functions
Hyperbolic functions are defined as:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) = \frac{1}{cosech\ x} \tag{28}$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{sech\ x} \tag{29}$$

whence

$$tanh\ x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1}{coth\ x} \ .$$

We readily find that $cosh\ z + sinh\ z = e^z$ and $cosh^2 z - sinh^2 z = 1$.

The derivatives of hyperbolic functions are easily calculated:

If $y = \sinh x = \frac{e^x - e^{-x}}{2}$, then

$$\frac{d \sinh x}{dx} = \frac{e^x + e^{-x}}{2} = \cosh x \ .$$

Similarly if $y = \cosh x$, $\frac{dy}{dx} = \sinh x$ and also if $y = \tanh x$, $\frac{dy}{dx} = sech^2 x$.

For inverse hyperbolic functions we take $y = \sinh^{-1} \frac{x}{a}$ as an example. If $y = \sinh^{-1} \frac{x}{a}$ then $x = a \sinh y$ and

$$\frac{dy}{dx} = \frac{1}{a \cosh y} = \frac{1}{a\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{x^2 + a^2}} \ .$$

Thus

$$\frac{d}{dx}\left(\sinh^{-1} \frac{x}{a}\right) = \frac{1}{\sqrt{x^2 + a^2}} \ .$$

If we have

$$y = \tanh^{-1}\frac{x}{a}, \quad \text{ie} \quad x = a\tanh y, \quad \frac{dx}{dy} = a\ sech^2 y \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{a}\frac{1}{\left(1-\tanh^2 y\right)} = \frac{a}{a^2 - x^2}\ .$$

<u>Parametric differentiation</u>

Often we have variables which are functions of a parameter, e.g. the time t: $x = x(t)$ and $y = y(t)$, but we need $\frac{dy}{dx}$. We make use of the chain rule and the derivative of an inverse function:

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}} \tag{30}$$

Example: The coordinate of a moving vehicle are given by $x = -t^2$, $y = (1/3)t^3$, where t is time. Find dy/dx when $t = 2$.

The direction at the point when $t = 2$ is given by the tangent to the trajectory at this point. Hence we need:

$$\left.\frac{dy}{dx}\right|_{t=2} \tag{31}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t^2}{-2t} = \frac{-t}{2} \tag{32}$$

which for $t = 2$ is

$$\left.\frac{dy}{dx}\right|_{t=2} = -1\ . \tag{33}$$

3.8 Stationary points

Derivatives give the rate of change of a function. An important application is finding the maximum and minimum of functions. If at some point, x_0 we have

$$f'(x_0) = 0$$

then this is a stationary point. For maxima and minima, $f'(x)$ changes sign around x_0. In summary, at a stationary point, where $f'(x_0) = 0$, we have three possibilities

1. Minimum: $f'(x_0) = 0$ and $f''(x_0) > 0$.
2. Maximum: $f'(x_0) = 0$ and $f''(x_0) < 0$.
3. Point of inflection: $f'(x_0) = 0$ and $f''(x_0) = 0$ and f'' changes sign through the point.

<u>Example</u>

Find the stationary point(s) of

$$f(x) = x\ln x$$

and determine its (their) nature.

$$f'(x) = \ln x + x\frac{1}{x} = \ln x + 1$$

we need to find x for which $f'(x) = 0$, i.e. $\ln x = -1$

$$x = e^{-1} = \frac{1}{e}.$$

To determine the nature of the stationary point we have to calculate the second derivative at $x_0 = e^{-1}$.

$$f''(x) = \frac{1}{x} \qquad \text{therefore} \qquad f''(x_0) = e > 0$$

Thus the stationary point is a minimum.

Example

Find the stationary points of $f(x) = x^4$.
we start by calculating the derivative:

$$f'(x) = 4x^3$$

thus $x = 0$ is a stationary point. To determine its nature, we calculate the second-order derivative

$$f''(x) = 12x^2 \qquad \text{thus} \qquad f''(0) = 0$$

However f'' does not change sign around $x = 0$. So I cannot conclude that $x = 0$ is a point of inflection. It is actually a minimum (prove it as an exercise).

3.9 Differentiation of a vector [see Riley, section 10.1]

The derivative of a vector function $\underline{a}(t)$ with respect to t is defined as

$$\frac{d\underline{a}}{dt} = \lim_{h \to 0} \frac{\underline{a}(t+h) - \underline{a}(t)}{h} \quad . \tag{34}$$

In cartesian coordinates:

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$

where a_x, a_y, a_z are functions of t. Then

$$\frac{d\underline{a}}{dt} = \frac{da_x}{dt}\underline{i} + \frac{da_y}{dt}\underline{j} + \frac{da_z}{dt}\underline{k} \ .$$

Some properties:

$$\frac{dc\underline{a}}{dt} = c\frac{d\underline{a}}{dt} \qquad (c \text{ is a constant})$$

$$\frac{d(\underline{a} + \underline{b})}{dt} = \frac{d\underline{a}}{dt} + \frac{d\underline{b}}{dt}$$

$$\frac{d(\underline{a}\cdot\underline{b})}{dt} = \underline{a}\cdot\frac{d\underline{b}}{dt} + \frac{d\underline{a}}{dt}\cdot\underline{b}$$

$$\frac{d(\underline{a}\times\underline{b})}{dt} = \underline{a}\times\frac{d\underline{b}}{dt} + \frac{d\underline{a}}{dt}\times\underline{b}$$

For example, if

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$$

is the position vector of a particle as a function of time, then the velocity and acceleration vectors are given by:

$$\underline{v} = \frac{d(\underline{r})}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k}$$

$$\underline{a} = \frac{d(\underline{v})}{dt} = \frac{d^2\underline{r}}{dt^2} = \frac{d^2x}{dt^2}\underline{i} + \frac{d^2y}{dt^2}\underline{j} + \frac{d^2z}{dt^2}\underline{k}$$

<u>Example</u>

Consider the motion of a particle in a circle at constant speed. Show that the velocity vector, $\underline{v}$, is perpendicular to the position vector $\underline{r}$ of the particle; that the acceleration vector is perpendicular to $\underline{v}$; and that

$$|\underline{a}| = \frac{|\underline{v}|^2}{|\underline{r}|}\ .$$

We notice that $\underline{r}$ and $\underline{v}$ are not constants, but their magnitudes are. So

$$|\underline{r}|^2 = \underline{r}\cdot\underline{r} = constant$$

$$|\underline{v}|^2 = \underline{v}\cdot\underline{v} = constant$$

This implies

$$\frac{d}{dt}(\underline{r}\cdot\underline{r}) = 0 \quad\Rightarrow\quad \frac{d\underline{r}}{dt}\cdot\underline{r} + \underline{r}\cdot\frac{d\underline{r}}{dt} = 0$$

or

$$\underline{r}\cdot\underline{v} = 0 \quad\Rightarrow\quad \underline{r},\ \underline{v}\ \text{ perpendicular}$$

In the same way:

$$\frac{d}{dt}(\underline{v}\cdot\underline{v}) = 0 \quad\Rightarrow\quad \frac{d\underline{v}}{dt}\cdot\underline{v} + \underline{v}\cdot\frac{d\underline{v}}{dt} = 0$$

or

$$\underline{v}\cdot\underline{a} = 0 \quad\Rightarrow\quad \underline{v},\ \underline{a}\ \text{ perpendicular}$$

Now we do

$$\frac{d}{dt}(\underline{r}\cdot\underline{v}) = \frac{d\underline{r}}{dt}\cdot\underline{v} + \underline{r}\cdot\frac{d\underline{v}}{dt} = \underline{v}\cdot\underline{v} + \underline{r}\cdot\underline{a} = 0$$

This implies that $\underline{r}$ and $\underline{a}$ are anti-parallel (they could only be parallel or anti-parallel from what demonstrated earlier), i.e. $\underline{r}\cdot\underline{a} = -|\underline{r}||\underline{a}|$. Therefore

$$0 = \underline{v}\cdot\underline{v} + \underline{r}\cdot\underline{a} = |\underline{v}|^2 - |\underline{r}||\underline{a}| \qquad\Rightarrow\qquad |\underline{a}| = \frac{|\underline{v}|^2}{|\underline{r}|}$$