2 Complex Numbers

2.1 The imaginary number i [see Riley 3.1, 3.3]

Complex numbers are a generalisation of real numbers. they occur in many branches of mathematics and have numerous applications in physics.

The imaginary number is

$$i = \sqrt{-1} \Leftrightarrow i^2 = -1$$

The obvious place to see where we have already needed this is in the solution to quadratic equation. Eg. finds the roots of

$$z^2 + 4z + 5 = 0$$

$$(z+2)^2 + 1 = 0$$

$$(z+2)^2 = -1$$

and

$$z_{1,2} = -2 \pm \sqrt{-1} \ .$$

So in this case we use the imaginary number and write the solutions as

$$z_{1,2} = -2 \pm i \ .$$

which is called a complex number. The general form of a complex number is

$$z = x + iy$$

where z is the conventional representation and is the sum of the real part x and i times the imaginary part y: these are denoted as

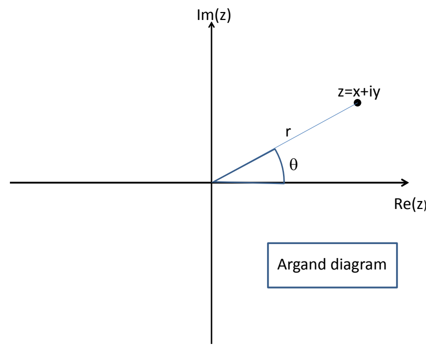
$$\begin{aligned} Re(z) &= x \\ Im(z) &= y \ , \end{aligned}$$

respectively. The imaginary or real part can be zero, so if the imaginary part is, the number is real and hence real numbers are just a subset of complex numbers.

Also when using the quadratic solutions formula, we had situations where there were no (real) roots as $b^2 - 4ac < 0$. we could have solved the above quadratic to get the same results:

$$z_{1,2} = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2\sqrt{-1}}{2} = -2 \pm i \ .$$

A complex number may also be written more compactly as $z = (x, y)$ where x and y are two real numbers which define the complex number and may be thought of as Cartesian coordinates.



Recall that in Cartesian coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Therefore we can represent z in polar coordinates as

$$z = x + iy = r(\cos\theta + i\sin\theta)\ .$$

The number r is called the modulus of z, written as $|z|$ or $\mathrm{mod}(z)$. This can be written in terms of x and y as

$$|z| = \sqrt{x^2 + y^2}\ .$$

The angle θ is called the argument of z, written as $\arg(z)$ (or $\arg z$) and is defined as

$$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)\ .$$

so $\arg(z)$ is the angle that the line joining the origin to z on an Argand diagram makes with the positive $x-$ axis. The anti-clockwise direction is taken to be positive by convention.

However, θ is not unique since $\theta + 2n\pi$ (n is zero or any integer) are also arguments for the same complex number. We therefore define a principal value of a complex number as that value of θ which satisfies $-\pi < \theta \leq \pi$. (it could also be $0 < \theta \leq 2\pi$). Also, account must be taken of the signs of x and y when determining in which quadrants $\arg(z)$ lies. E.g. if x and y are both negatives, then $-\pi < \arg(z) < -\pi/2$ rather than $0 \leq arg(z) < \pi/2$ even though the ratios of x and y will be the same when both negative or both positive.

Example

Find the modulus and argument of $z = -3 + 5i$.

$$|z| = \sqrt{3^2 + 5^2} = \sqrt{34}$$

$$\arg(z) = \tan^{-1}\frac{5}{-3} = -1.03 \quad \text{or} \; -1.03 + \pi = 2.11$$

but given where z must lie on the Argand diagram:

$$\arg(z) = 2.11$$

i.e. positive and $\leq \pi$.

Note: for $z = 3 - 5i$ we would have had $\arg(z) = -1.03$.

2.2 Operations with complex numbers [see Riley 3.2]

Addition and subtraction

The addition or subtraction of two complex numbers leads to, in general, another complex number where the real and imaginary components are added separately. Therefore for two complex numbers z_1 and z_2:

$$z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2) \; .$$

Note that complex numbers, as with real numbers, satisfy the commutative and associative laws of addition:

$$z_1 + z_2 = z_2 + z_1$$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \; .$$

Multiplication

Multiplication of two complex complex numbers gives, in general, another complex number. The product is calculated by multiplying out in full.

$$\begin{aligned} z_1 \; z_2 &= (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

Multiplication is both commutative and associative

$$z_1 \; z_2 = z_2z_1$$

$$(z_1z_2)z_3 = z_1(z_2z_3)$$

and also has the simple properties

$$|z_1z_2| = |z_1||z_2|$$

$$\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$$

e.g. for $z_1 = 5 - 3i$, $z_2 = 1 + 2i$:

$$\begin{aligned} |z_1| &= \sqrt{5^2 + (-3)^2} = \sqrt{34} \\ |z_2| &= \sqrt{1^2 + 2^2} = \sqrt{5} \end{aligned}$$

and

$$|z_1 z_2| = \sqrt{(5+6)^2 + (10-3)^2} = \sqrt{170} = \sqrt{34 \times 5} = |z_1||z_2|$$

Complex conjugate

If we define a complex number

$$z = x + iy \ ,$$

then its complex conjugate is

$$z^* = x - iy \ .$$

So the complex conjugate has the same magnitude as z and when multiplied by z gives a real positive result:

$$\begin{aligned} zz^* &= (x+iy)(x-iy) \\ &= x^2 - ixy + ixy - i^2y^2 \\ &= x^2 + y^2 = |z|^2 \end{aligned}$$

Likewise for any two complex numbers

$$|z_1 z_2|^2 = z_1 z_2 z_1^* z_2^* = z_1 z_1^* z_2 z_2^* = |z_1|^2 |z_2|^2$$

and since all moduli are positive

$$|z_1 z_2| = |z_1||z_2|$$

as stated before. Also

$$\begin{aligned} z + z^* &= (x+iy) + (x-iy) = 2x = 2Re(z) \\ z - z^* &= (x+iy) - (x-iy) = 2iy = 2iIm(z) \end{aligned}$$

Note that no matter how complicated the expression we can always form the conjugate by replacing every i by $-i$.

Example

Consider

$$z = w^{(3y+2ix)}$$

where

$$w = x + 5i \ .$$

So

$$z = (x+5i)^{3y+2ix}$$

and

$$z^* = (x-5i)^{3y-2ix}$$

Division

What is

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \ .$$

To evaluate, we multiply top and bottom by the complex conjugate of the denominator, z_2^*:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{x_1+iy_1}{x_2+iy_2}\frac{x_2-iy_2}{x_2-iy_2} = \frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2} \\ &= \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+i\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}\end{aligned}$$

and so this multiplication allowed us to separate out real and imaginary components. So in brief:

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{|z_2|^2} \ .$$

<u>Example</u>

$$\frac{-7+3i}{4+i} = \frac{(-7+3i)(4-i)}{(4+i)(4-i)} = \frac{-28+7i+12i+3}{16+1} = -\frac{25}{17}+\frac{19}{17}i$$

Division also has some simple properties:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1)-\arg(z_2)$$

2.3 Exponential form for complex numbers and Euler's equation [see Riley 3.3]

We have already defined

$$z = x+iy$$

and seen that this can be written as

$$z = r(\cos\theta + i\sin\theta) \ .$$

Another form of a complex number, which will allow various operations to be performed far more easily, uses Euler's equation:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

(this will be proved later, once series are introduced). Therefore we can express z as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} \ .$$

Also

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

and

$$z^* = re^{-i\theta} \ .$$

We again associate r with $|z|$ and θ with $\arg(z)$ and note that rotation by θ is the same as rotation by $\theta+2n\pi$ where n is any integer:

$$re^{i\theta} = re^{i(\theta+2n\pi)} \ .$$

Example

Write

$$z = (4+3i)e^{i\pi/3}$$

in the form $x+iy$ (x, y real).
We first expand the exponent:

$$e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

and then multiply:

$$z = \frac{1}{2}(1+i\sqrt{3})(4+3i) = \frac{4-3\sqrt{3}}{2} + i\frac{3+4\sqrt{3}}{2}\ .$$

Multiplication and division become more simple when using this exponential form.

For

$$z_1 = r_1 e^{i\theta_1} \qquad \text{and} \qquad z_2 = r_2 e^{i\theta_2}:$$

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

and

$$|z_1 z_2| = |z_1||z_2|$$

and

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

follow immediately.
In the same way:

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_1 e^{i\theta_1}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

and

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

follow immediately.

Example

Considering the real and imaginary parts of the product

$$e^{i\theta}e^{i\phi}$$

prove the standard formulae for $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$.

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$$

and

$$
\begin{aligned}
e^{i\theta}e^{i\phi} &= (\cos(\theta) + i\sin(\theta))(\cos(\phi) + i\sin(\phi)) \\
&= \cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi - \sin\theta\sin\phi \qquad (1)
\end{aligned}
$$

as the two above expresssions for $e^{i\theta}e^{i\phi}$ must be equal, we can equate the real parts and imaginary parts. We obtain in this way

$$
\begin{aligned}
\cos(\theta + \phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\
\sin(\theta + \phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi
\end{aligned}
$$

2.4 Hyperbolic and trigonometric functions and complex numbers [see Riley 3.7]

Given our form of exponential representation of a complex number in polar coordinates, we can find new expressions for the cosine and sine of a quantity. We have

$$
\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta \\
e^{-i\theta} &= \cos\theta - i\sin\theta
\end{aligned}
$$

from which we derive

$$
\begin{aligned}
e^{i\theta} + e^{-i\theta} &= 2\cos\theta \Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \\
e^{i\theta} - e^{-i\theta} &= 2i\sin\theta \Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}
\end{aligned} \qquad (2)
$$

Hyperbolic functions are defined as:

$$
\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\cosh x &= \frac{e^x + e^{-x}}{2} \; .
\end{aligned}
$$

So there are simple relationships between the trigonometric and hyperbolic functions:

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \cos x$$

or

$$\cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x \; .$$

And, in the same way:

$$\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2} = i\sin x$$

or

$$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} = i\sinh x \; .$$

2.5 de Moivre's theorem and application [see Riley, section 3.4]

Since we have

$$\left(e^{i\theta}\right)^n = e^{in\theta} \ ,$$

then

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \ .$$

This is the Moivre's theorem. It is valid for all n; real, imaginary, or complex.

Trigonometric identities

We can express a multiple-angle function in terms of a polynomial of single angles and e.g. derive identities, e.g. $\cos(2x)$:

$$\cos(2x) + i\sin(2x) = (\cos(x) + i\sin(x))^2 = \cos^2 x - \sin^2 x + 2i\sin x\cos x \ .$$

Equating real and imaginary parts

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ \sin(2x) &= 2\sin x\cos x \end{aligned}$$

Identities for $z = e^{i\theta}$:

$$\begin{aligned} z^n + \frac{1}{z^n} &= 2\cos n\theta \\ z^n - \frac{1}{z^n} &= 2i\sin n\theta \end{aligned}$$

which can be obtained from de Moivre's theorem and we have already shown for the case of $n = 1$.

Finding roots

To find n complex roots of e.g. $z^n - 1 = 0$, where n is a positive integer, we use the form

$$1 = e^{2\pi ki} = \cos 2\pi k + i\sin 2\pi k$$

where k is any integer or 0. Thus

$$z = 1^{1/n} = (\cos 2\pi k + i\sin 2\pi k)^{1/n} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$$

which letting $k = 0, 1, 2, 3, ..., (n-1)$ has n distinct values $z_1, z_2, ..., z_n$.

Example

Find the roots $(-8)^{1/3}$.

There is one obvios value, -2, but there are two other complex roots. we can write

$$-8 = 8e^{i(\pi + 2n\pi)}$$

thus

$$(-8)^{1/3} = 2e^{(i\pi/3)(2n+1)} \ .$$

So the roots are for $n = 0, 1, 2$:

$$\begin{aligned} z_1 &= 2e^{i\pi/3} = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 1 + i\sqrt{3} \\ z_2 &= 2e^{i\pi} = 2(\cos\pi + i\sin\pi) = -2 \\ z_3 &= 2e^{i5\pi/3} = 2(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}) = 1 - i\sqrt{3} \end{aligned}$$