1 Vectors

1.1 Scalars and vectors [Riley 7.1]

Scalars: These are the simplest kind of physical quantity that can be completely specified by its magnitude, a single number together with the units in which they are measured. Examples include temperature, time, density, etc.

Vectors: A quantity that requires both a magnitude and a direction in space to specify it completely. Examples include force, velocity, electric field, etc.

1.2 Addition and subtraction of vectors [7.2]

The vector sum

$$\underline{c} = \underline{a} + \underline{b}$$

of two displacement vectors is the displacement vector that results from performing first one then the other displacement.
Vector addition is commutative

$$\underline{c} = \underline{a} + \underline{b} = \underline{b} + \underline{a} \; .$$

When adding three vectors, this leads to the associativity property of addition, i.e.

$$\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$$

In fact, in general, it is immaterial in what order any number of vectors are added.
The subtraction of two vectors is very similar to their addition:

$$\underline{a} - \underline{b} = \underline{a} + (-\underline{b})$$

where $-\underline{b}$ is a vector of equal magnitude but exactly the opposite direction to $\underline{b}$.
The subtraction of two equal vectors yields the zero vector, $\underline{0}$, which has zero magnitude and no associated direction.

1.3 Multiplication of a vector by a constant [7.3]

Multiplication of a vector by a scalar changes the magnitude but not the direction, although if the scalar is negative, we obtain a vector pointing in the opposite direction. Multiplication by a scalar is associative, commutative and distributive over addition. Therefore for arbitrary vectors $\underline{a}$ and $\underline{b}$ and arbitrary scalars λ and μ:

$$(\lambda\mu)\underline{a} = \lambda(\mu\underline{a}) = \mu(\lambda\underline{a}) \qquad \text{[commutative]}$$

$$\lambda(\underline{a} + \underline{b}) = \lambda\underline{a} + \lambda\underline{b} \qquad \text{[distributive]}$$

$$(\lambda + \mu)\underline{a} = \lambda\underline{a} + \mu\underline{a} \qquad \text{[associative]}$$

1.4 Basis vector, position vector and unit vector [Riley 7.4]

Given any three different vectors $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, which do not all lie in a plane, we can, in 3-D space, write any other vector in terms of scalar multiples of them.

$$\underline{a} = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3 \ .$$

The vectors $\hat{\underline{e}}_1$, $\hat{\underline{e}}_2$, $\hat{\underline{e}}_3$ are said to form a basis (for the 3-D space).The scalars which may be positive, negative or zero are called the components of the vector $\underline{a}$ with respect to this basis.

In the Cartesian coordinate system (x, y, z) we introduce the unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$ which point along the positive $x-$, $y-$ and $z-$ axis respectively. A vector $\underline{a}$ may then be written as the sum of three vectors

$$\underline{a} = a_x\hat{\underline{i}} + a_y\hat{\underline{j}} + a_k\hat{\underline{k}} \ .$$

or in short (a_x, a_y, a_k). And the basis vectors may be represented by $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ for $\underline{i}$, $\underline{j}$ and $\underline{k}$ respectively. These are therefore called unit vectors.

A special case of the general vector $\underline{a}$ is the position vector which starts at the origin to the point (x, y, z):

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} \ .$$

To add and subtract vectors, we just add/subtract the components:

$$\begin{aligned}
\underline{a} + \underline{b} &= (a_x\underline{i} + a_y\underline{j} + a_z\underline{k}) + (b_x\underline{i} + b_y\underline{j} + b_z\underline{k}) = (a_x + b_x)\underline{i} + (a_y + b_y)\underline{j} + (a_z + b_z)\underline{k} \\
\underline{a} - \underline{b} &= (a_x\underline{i} + a_y\underline{j} + a_z\underline{k}) - (b_x\underline{i} + b_y\underline{j} + b_z\underline{k}) = (a_x - b_x)\underline{i} + (a_y - b_y)\underline{j} + (a_z - b_z)\underline{k}
\end{aligned}$$

Multiplication by a scalar leads to multiplication of each component

$$\lambda\underline{a} = \lambda a_x\underline{i} + \lambda a_y\underline{j} + \lambda a_z\underline{k}$$

1.5 Magnitude of a vector [See Riley, Sec. 7.5]

The magnitude of a vector $\underline{a}$ is denoted by $|\underline{a}|$ (or a) and is the "length" of the vector (in the units of the physical quantity that a represents).

For our general vector a,

$$|\underline{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Example

Two particles have velocities $\underline{v}_1 = 1\underline{i} + 3\underline{j} + 6\underline{k}$ and $\underline{v}_2 = 1\underline{i} - 2\underline{k}$. Find the velocity $\underline{u}$ of the second particle relative to the first.

$$\underline{u} = \underline{v}_2 - \underline{v}_1 = (1 - 1)\underline{i} + (0 - 3)\underline{j} + (-2 - 6)\underline{k} = -3\underline{j} - 8\underline{k}$$

and

$$|\underline{u}| = \sqrt{(-3)^2 + (-8)^2}$$

In general a vector whose magnitude equals unity is called a unit vector. The unit vector in the direction $\underline{a}$ is

$$\hat{\underline{a}} = \hat{\underline{e}}_a = \frac{\underline{a}}{|\underline{a}|} \ .$$

Note if we have a vector $\lambda \hat{\underline{e}}_a$, then we have the magnitude and direction explicitly separated.

<u>Example</u>

A point P divides a line segment AB in the ratio $\lambda : \mu$. If the position vectors of the points A and B are $\underline{a}$ and $\underline{b}$, respectively, find the position vector of point P.

The vector connecting $\underline{a}$ and $\underline{b}$ is

$$\underline{AB} = \underline{b} - \underline{a} \ .$$

Note distances:

$$\frac{BP}{AB} = \frac{\mu}{\mu + \lambda}, \qquad \frac{AP}{AB} = \frac{\lambda}{\mu + \lambda}$$

First consider going from O to A and then A to P:

$$\begin{aligned} \underline{OP} &= \underline{a} + \frac{\lambda}{\mu + \lambda} \underline{AB} \\ &= \underline{a} + \frac{\lambda}{\mu + \lambda} (\underline{b} - \underline{a}) \\ &= \left(1 - \frac{\lambda}{\mu + \lambda}\right) \underline{a} + \frac{\lambda}{\mu + \lambda} \underline{b} \\ &= \frac{\mu}{\mu + \lambda} \underline{a} + \frac{\lambda}{\mu + \lambda} \underline{b} \end{aligned}$$

1.6 The scalar (or dot) product [Riley 7.6.1]

As the name suggests, the product yields a scalar quantity. It is defined as follows:

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta \qquad (0 \leq \theta \leq \pi)$$

where θ is the angle between the two vectors $\underline{a}$ and $\underline{b}$.
It follows that two non-zero vecors are perpendicular if

$$\underline{a} \cdot \underline{b} = 0 \ .$$

And, for the Cartesian unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ we have:

$$\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1$$

and

$$\underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{k} = \underline{k} \cdot \underline{i} = 0 \ .$$

From these relations, we can then write the scalar product of two vectors, $\underline{a}$ and $\underline{b}$, in terms of the components:

$$\underline{a} \cdot \underline{b} = (a_x\underline{i} + a_y\underline{j} + a_z\underline{k}) \cdot (b_x\underline{i} + b_y\underline{j} + b_z\underline{k}) = a_xb_x + a_yb_y + a_zb_z \ .$$

From the above it follows that:

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \frac{a_xb_x + a_yb_y + a_zb_z}{|\underline{a}||\underline{b}|}$$

Also the magnitude of a vector can be found from the scalar product

$$\underline{a} \cdot \underline{a} = |\underline{a}||\underline{a}|\cos 0 = |\underline{a}|^2 = a_x^2 + a_y^2 + a_z^2 \ .$$

We can also get the cosine rule from the scalar product. Let

$$\underline{c} = \underline{a} + \underline{b}$$

then

$$\begin{aligned}
\underline{c} \cdot \underline{c} &= |\underline{a}|^2 + |\underline{b}|^2 + \underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{a} \\
&= |\underline{a}|^2 + |\underline{b}|^2 + 2|\underline{a}||\underline{b}|\cos\alpha \\
&= |\underline{a}|^2 + |\underline{b}|^2 - 2|\underline{a}||\underline{b}|\cos\beta
\end{aligned}$$

Example

Find the angle between the vectors $\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$ and $\underline{b} = 2\underline{i} + 3\underline{j} + 4\underline{k}$.

We need to calculate:

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$$

$$\underline{a} \cdot \underline{b} = 1 \times 2 + 2 \times 3 + 3 \times 4 = 20$$

$$|\underline{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$|\underline{b}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

From which

$$\cos\theta = \frac{20}{\sqrt{14}\sqrt{29}} \simeq 0.9926 \Rightarrow \theta \simeq 0.12 \text{ rad.}$$

Note that as we are dealing with just a magnitude the following properties hold:

$$\underline{a} \cdot \underline{b} = \underline{a} \cdot \underline{b} \qquad \text{[commutative]}$$

$$\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} \qquad \text{[distributive]}$$

$$(\lambda\underline{a}) \cdot (\mu\underline{b}) = \lambda\mu(\underline{a} \cdot \underline{b}) \qquad [\lambda, \mu \quad \text{scalars}]$$

1.7 The vector (or cross) product [see Riley Section 7.6.2]

The vector product is defined as follows:

$$\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin\theta \hat{\underline{n}}$$

where the magnitude is $|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}| \sin\theta$ and $\hat{\underline{n}}$ is a direction perpendicular to the plane defined by $\underline{a}$ and $\underline{b}$.

The direction of $\hat{\underline{n}}$ is given by the right-hand rule: if your index finger points in the direction of $\underline{a}$ and your middle finger in the direction of $\underline{b}$, then your thumb gives the direction of $\hat{\underline{n}}$.

The vector product is distributive over addition, but anti-commutative and non-associative:

$$(\underline{a} + \underline{b}) \times \underline{c} = (\underline{a} \times \underline{c}) + (\underline{b} \times \underline{c})$$

$$(\underline{b} \times \underline{a}) = -(\underline{a} \times \underline{b})$$

$$\underline{a} \times (\underline{b} \times \underline{c}) \neq (\underline{a} \times \underline{b}) \times \underline{c}$$

Also, if two vectors are non-zero, then if $\underline{a} \times \underline{b} = 0$, then $\underline{a}$ is parallel (antiparallel) to $\underline{b}$. And $\underline{a} \times \underline{a} = 0$.

For the unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ we have,

$$\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = \underline{0}$$

$$\underline{i} \times \underline{j} = -\underline{j} \times \underline{i} = \underline{k}$$

$$\underline{j} \times \underline{k} = -\underline{k} \times \underline{j} = \underline{i}$$

$$\underline{k} \times \underline{i} = -\underline{i} \times \underline{k} = \underline{j}$$

Therefore, for general vectors $\underline{a}$, $\underline{b}$ in terms of the components with respect to the basis set $\underline{i}$, $\underline{j}$, $\underline{k}$:

$$\begin{aligned}
\underline{a} \times \underline{b} &= (a_x\underline{i} + a_j\underline{j} + a_z\underline{k}) \times (b_x\underline{i} + b_y\underline{j} + b_z\underline{k}) \\
&= a_x b_y(\underline{i} \times \underline{j}) + a_x b_z(\underline{i} \times \underline{k}) + a_y b_x(\underline{j} \times \underline{i}) \\
&+ a_y b_z(\underline{j} \times \underline{k}) + a_z b_x(\underline{k} \times \underline{i}) + a_z b_y(\underline{k} \times \underline{j}) \\
&= (a_x b_y - a_y b_x)\underline{k} + (a_z b_x - a_x b_z)\underline{j} + (a_y b_z - a_z b_y)\underline{i}
\end{aligned}$$

We notice that the magnitude of the cross product is the area of a parallelogram. In fact the area of a parallelogram of sides $\underline{a}$ and $\underline{b}$ is

$$|\underline{a}|h = |\underline{a}||\underline{b}| \sin\theta = |\underline{a} \times \underline{b}| \ .$$

1.8 Triple products [see Riley, Section 7.6.3 and 7.6.4]

We have the scalar triple product and the vector triple product.

Scalar triple product

This is the dot product of a vector $\underline{a}$ with the crossed product formed from two other vectors $\underline{b}$ and $\underline{c}$, i.e.

$$\underline{a} \cdot (\underline{b} \times \underline{c}) \ ,$$

with the result a number.

Expressed in terms of the components of each vector with respect the Cartesian basis set, the scalar triple product is

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = a_x(b_y c_z - b_z c_y) - a_y(b_x c_z - b_z c_x) + (b_x c_y - b_y c_x)a_z \ .$$

Using the notation

$$[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot (\underline{b} \times \underline{c}) \ ,$$

then

$$[\underline{a}, \underline{b}, \underline{c}] = [\underline{b}, \underline{c}, \underline{a}] = [\underline{c}, \underline{a}, \underline{b}] = -[\underline{a}, \underline{c}, \underline{b}] = -[\underline{b}, \underline{a}, \underline{c}] = -[\underline{c}, \underline{b}, \underline{a}]$$

i.e. the scalar triple product is unchanged under cyclic permutation of the vector $\underline{a}$, $\underline{b}$, $\underline{c}$. Other permutations give the negative of the original product.

The scalar triple product gives the volume of a parallelepiped.

Consider the parallelepiped defined by the three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$. The vector $\underline{v} = \underline{a} \times \underline{b}$ is perpendicular to the base and has magnitude $ab \sin\theta$, i.e. the area of the base parallelogram. Also, $\underline{v} \cdot \underline{c} = vc \cos\phi$, where ϕ is the angle between $\underline{c}$ and the perpendicular to the base. But as $c \cos\phi$ is the vertical height of the parallelepiped, then $(\underline{a} \times \underline{b}) \cdot \underline{c}$ is the area of the base multiplied by the perpendicular height, i.e. the volume.

Example

Find the volume V of a parallelepiped with sides $\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$, $\underline{b} = 4\underline{i} + 5\underline{j} + 6\underline{k}$ and $\underline{c} = 7\underline{i} + 8\underline{j} + 10\underline{k}$.

We first calculate

$$\underline{a} \times \underline{b} = -3\underline{i} + 6\underline{j} - 3\underline{k}$$

from which

$$V = |\underline{a} \cdot (\underline{b} \times \underline{c})| = |(\underline{a} \times \underline{b}) \cdot \underline{c}| = |(-3\underline{i} + 6\underline{j} - 3\underline{k}) \cdot (7\underline{i} + 8\underline{j} + 10\underline{k})| = |(-3 \cdot 7 + 6 \cdot 8 + (-3) \cdot 10| = 3$$

Vector triple product

The product $\underline{a} \times (\underline{b} \times \underline{c})$ is perpendicular to $\underline{a}$ and lies in the plane containing $\underline{b}$ and $\underline{c}$. It can be expressed in terms of them, i.e.

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c} \ ,$$

$$(\underline{a} \times \underline{b}) \times \underline{c} = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} \cdot \underline{c})\underline{a} \ .$$

Example

Show the Lagrange identity

$$(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c})$$

First, we treat the LHS as a scalar triple product of $\underline{a} \times \underline{b}$, $\underline{c}$ and $\underline{d}$. Then we cycle.

$$(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d}) = \underline{d} \cdot [(\underline{a} \times \underline{b}) \times \underline{c}]$$

Using the result

$$(\underline{a} \times \underline{b}) \times \underline{c} = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} \cdot \underline{c})\underline{a}$$

we can then derive

$$(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d}) = \underline{d} \cdot [(\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} \cdot \underline{c})\underline{a}] = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c})$$

1.9 Equations of lines, planes and spheres [see Riley, Sec 7.7]

Equation of a line

The vector $\underline{\mathrm{b}}$ is in the direction AR and $\lambda\underline{b}$ is the vector from A to R. From this diagram we can see that

$$\underline{r} = \underline{a} + \lambda\underline{b} \ .$$

Writing this in component form,

$$\begin{aligned} x\underline{i} &= a_x\underline{i} + \lambda b_x\underline{i} \\ y\underline{j} &= a_y\underline{j} + \lambda b_y\underline{j} \\ z\underline{k} &= a_z\underline{k} + \lambda b_z\underline{k} \end{aligned}$$

we can also write the equation of a line as:

$$\frac{x - a_x}{b_x} = \frac{y - a_y}{b_y} = \frac{z - a_z}{b_z} = \lambda \ .$$

Additionally, we can take the vector product of $\underline{r}$ with $\underline{b}$. As $\underline{b} \times \underline{b} = \underline{0}$ we obtain

$$(\underline{r} - \underline{a}) \times \underline{b} = \underline{0}$$

which is another form.

Finally, the equation of a line going through two fixed points A and C with position vectors $\underline{a}$ and $\underline{c}$. Since AC is given by $\underline{c} - \underline{a}$, the position vector of a general point on the line is:

$$\underline{r} = \underline{a} + \lambda(\underline{c} - \underline{a}) \ .$$

We now introduce the useful concept of *direction cosines*. For a vector $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, the position vector $\underline{r}$ has three angles which are from the three axes x, y, z. The components and angles can be related by

$$\begin{aligned} x &= |\underline{r}| \cos\theta_x, \\ y &= |\underline{r}| \cos\theta_y, \\ z &= |\underline{r}| \cos\theta_z, \end{aligned}$$

from which:

$$\frac{\underline{r}}{|\underline{r}|} = \hat{\underline{r}} = \underline{i}\cos\theta_x + \underline{j}\cos\theta_y + \underline{k}\cos\theta_z$$

where the cosines of the three angles are called the direction cosines.

Example

The line through the point $(2, 1, 5)$ with direction cosines $(l, m, n) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is represented by the equation

$$\underline{r} = \underline{r}_1 + \lambda\hat{\underline{u}}$$

where $\underline{r}_1 = 2\underline{i} + \underline{j} + 5\underline{k}$ and $\hat{\underline{u}} = \frac{1}{\sqrt{3}}\underline{i} + \frac{1}{\sqrt{3}}\underline{j} + \frac{1}{\sqrt{3}}\underline{k}$. The equation of the line can also be written as

$$\frac{x-2}{\frac{1}{\sqrt{3}}} = \frac{y-1}{\frac{1}{\sqrt{3}}} = \frac{z-5}{\frac{1}{\sqrt{3}}} = \lambda \ .$$

We can absorb the $\sqrt{3}$ into λ:

$$x - 2 = y - 1 = 5 = \lambda' \ .$$

Equation of a plane

We examine now the equation of a plane, passing through point A, of position vector $\underline{a}$, and with unit vector $\hat{\underline{n}}$ normal to the plane.

From the diagram:

$$\underline{r} = \underline{\rho} + \underline{a}$$

or

$$\underline{r} - \underline{a} = \underline{\rho} \ .$$

And taking the scalar product with $\hat{\underline{n}}$:

$$(\underline{r} - \underline{a}) \cdot \hat{\underline{n}} = \underline{\rho} \cdot \hat{\underline{n}} \ .$$

which becomes, as $\underline{\rho}$ and $\hat{\underline{n}}$ are perpendicular

$$(\underline{r} - \underline{a}) \cdot \hat{\underline{n}} = 0 \ .$$

We can re-write the above as

$$\underline{r} \cdot \hat{\underline{n}} = \underline{a} \cdot \hat{\underline{n}} = d$$

where $d = \underline{a} \cdot \hat{\underline{n}}$ is the perpendicular distance of the plane from the origin. For a unit vector

$$\hat{\underline{n}} = l\underline{i} + m\underline{j} + n\underline{k}$$

we have the equation in component form:

$$\boxed{lx + my + nz = d \ .}$$

This is the definition of a plane when we have a point in the plane and a vector perpendicular to the plane.

We can also define a plane by three points which are contained within it. For three points given by position vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ we have

$$\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a}) + \mu(\underline{c} - \underline{a}) \ .$$

Here $\underline{a}$ is the starting point and all other points on the plane may be reached by defining two non-parallel directions, e.g. $\underline{b} - \underline{a}$ and $\underline{c} - \underline{a}$.

Example

Find the direction of the line of intersection of the two planes $x + 3y - z = 5$ and $2x - 2y + 4z = 3$.

As above, for a plane given by $ax + by + cz = d$, then the vector (a, b, c) is perpendicular to the plane. Therefore the normal vectors are:

$$\underline{n}_1 = \underline{i} + 3\underline{j} - \underline{k}$$

and

$$\underline{n}_2 = 2\underline{i} - 2\underline{j} + 4\underline{k} \ .$$

The line of intersection $\underline{\rho}$ must be parallel to both planes and thus perpendicular to both normals:

$$\underline{\rho} = \underline{n}_1 \times \underline{n}_2 = 10\underline{i} - 6\underline{j} - 8\underline{k} \ .$$

Equation of a sphere

This can be simply expressed as all points on a sphere are equidistant from a fixed point, with the distance equal to the radius:

$$|\underline{r} - \underline{c}|^2 = (\underline{r} - \underline{c}) \cdot (\underline{r} - \underline{c}) = a^2$$

where $\underline{c}$ is the position vector of the centre and a is the radius of the sphere.

1.10 Distances using vectors [see Riley, section 7.8]

Distance from a point to a line

A line has direction $\underline{b}$ and passes through a point A whose position vector is $\underline{a}$. We want to find the minimum distance d of the line from a point P.

We can see that

$$d = |\underline{\rho} - \underline{a}| \sin\theta$$

which can be re-written in terms of the vector product as

$$d = |(\underline{\rho} - \underline{a}) \times \hat{\underline{b}}|$$

where

$$\hat{\underline{b}} = \frac{\underline{b}}{|\underline{b}|}$$

is the unit vector along the line.

Example

Find the minimum distance from the point P with coordinates $(1, 2, 1)$ to the line $\underline{r} = \underline{a} + \lambda\underline{b}$ where $\underline{a} = \underline{i} + \underline{j} + \underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + 3\underline{k}$.

The line passes through $(1, 1, 1)$ and has direction $2\underline{i} - \underline{j} + 3\underline{k}$. The unit vector in this direction is

$$\hat{\underline{b}} = \frac{1}{\sqrt{14}}(2\underline{i} - \underline{j} + 3\underline{k}) \ .$$

The position vector $\underline{\rho}$ of P is

$$\underline{\rho} = \underline{i} + 2\underline{j} + \underline{k} \ .$$

Thus

$$(\underline{\rho} - \underline{a}) \times \hat{\underline{b}} = \frac{1}{\sqrt{14}}(3\underline{i} - 2\underline{k}) \ .$$

and

$$d = \sqrt{\frac{13}{14}}$$

Distance from a point to a plane

To find the minimum distance, d, from a point P (with position vector $\underline{p}$) to the plane defined by a point A (with position vector $\underline{a}$) and unit vector $\hat{\underline{n}}$ perpendicular to the plane:

$$d = |(\underline{a} - \underline{p}) \cdot \hat{\underline{n}}|$$

Example

Find the distance from the point P(1,2,3) to the plane that contains the points A(0,1,0), B(2,3,1) and C(5,7,2), with position vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$.

Two vectors in the plane are

$$\underline{b} - \underline{a} = 2\underline{i} + 2\underline{j} + \underline{k}$$

and

$$\underline{c} - \underline{a} = 5\underline{i} + 6\underline{j} + 2\underline{k}$$

and so a normal to the plane is

$$\underline{n} = (\underline{b} - \underline{a}) \times (\underline{c} - \underline{a}) = -2\underline{i} + \underline{j} + 2\underline{k}$$

and

$$\hat{\underline{n}} = \frac{1}{3}(-2\underline{i} + \underline{j} + 2\underline{k}) \ .$$

Thus

$$d = |(\underline{a} - \underline{p}) \cdot \hat{\underline{n}}| = |(-\underline{i} - \underline{j} - 3\underline{k}) \cdot \frac{1}{3}(-2\underline{i} + \underline{j} + 2\underline{k})| = |-\frac{5}{3}| = \frac{5}{3} \ .$$

1.11 Other systems of coordinates: polar coordinates in two dimensions

So far we only used a cartesian system of coordinates. We notice here that other systems of coordinates are commonly used, for example polar cordinates in two dimensions.

In two dimensions the position vector is

$$\underline{r} = x\underline{i} + y\underline{j} \ .$$

The transformation to polar coordinates is defined by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \ . \end{aligned}$$

In polar coordinates the position vectors is identified by the coordinates (r, θ).

Polar coordinates in two dimensions, as well as other systems of coordinates, will be examined more in detail at the end of the course.