RELATIVITY (MTH6132)

SOLUTIONS TO THE PROBLEM SET 4

1. For the particles in the problem one has

$$\begin{aligned}
\bar{p} &= m(1,0,0,0),\\
\bar{p}_1 &= m_1\gamma_1(1,-u_1,0,0),\\
\bar{p}_2 &= m_2\gamma_2(1,u_2,0,0).
\end{aligned}$$

In particular, notice the minus sign on $\bar{p}_1$ written down for convenience. Conservation of momentum reads

$$\bar{p} = \bar{p}_1 + \bar{p}_2.$$

Equating components one finds

$$\begin{aligned}
m &= m_1\gamma_1 + m_2\gamma_2,\\
0 &= -m_1\gamma_1 u_1 + m_2\gamma - 2u_2.
\end{aligned}$$

From the last equation one obtains

$$m_1\gamma_1 u_1 = m_2\gamma_2 u_2. \tag{1}$$

Squaring the equation of conservation of momentum

$$|\bar{p}|^2 = |\bar{p}_1|^2 + |\bar{p}_2|^2 + 2\bar{p}_1\cdot\bar{p}_2.$$

Recall that

$$|\bar{p}|^2 = -m^2, \quad |\bar{p}_1|^2 = -m_1^2, \quad |\bar{p}_2|^2 = -m_2^2$$

and that

$$\bar{p}_1\cdot\bar{p}_2 = m_1 m_2\gamma_1\gamma_2(-1-u_1u_2).$$

The minus sign in $-u_1u_2$ comes from the minus sign in $\bar{p}_1$. All this gives

$$m^2 = m_1^2 + m_2^2 + 2m_1m_2\gamma_1\gamma_2(1+u_1u_2).$$

One can use (1) to eliminate $m_1\gamma_1$. Hence,

$$\begin{aligned}
m^2 &= m_1^2 + m_2^2 + 2m_2^2\gamma_2^2\frac{u_2}{u_1}(1+u_1u-2),\\
&= m_1^2 + m_2^2 + 2m_2^2\frac{u^2}{1-u^2}\left(\frac{1}{u_1}+u_2\right).
\end{aligned}$$

2. In this problem one has

$$\begin{aligned}
\bar{p} &= m_0(1,0,0,0),\\
\bar{p}_\nu &= (h\nu, h\nu, 0, 0),\\
\bar{p}' &= m'\gamma'(1,u',0,0).
\end{aligned}$$

At the begining one has two particles and at the end one. The conservation of momentum reads

$$\bar{p} + \bar{p}_\nu = \bar{p}'.$$

Squaring the above expression:

$$|\bar{p}|^2 + |\bar{p}_\nu|^2 + 2\bar{p}\cdot\bar{p}_\nu = |\bar{p}'|^2.$$

Using that

$$|\bar{p}|^2 = -m_0^2, \quad |\bar{p}'|^2 = -m'^2, \quad \bar{p} \cdot \bar{p}_\nu = -m_0 h\nu,$$

so that one obtains

$$m' = \sqrt{m_0^2 + 2m_0 h\nu}.$$

In order to obtain the velocity one needs to equate the components in th equation of momentum. One has that

$$m'\gamma' = m_0 + h\nu, \quad m'\gamma' u' = h\nu.$$

Dividing the second equation by the first one obtains:

$$u' = \frac{h\nu}{m_0 + h\nu}.$$

3. In this problem before the collision one has that

$$\bar{p}_1 = m_1\gamma_1(1, u_1, 0, 0) = (E_1, u_1 E_1, 0, 0),$$
$$\bar{p}_2 = (m_2, 0, 0, 0).$$

After the collision

$$\bar{p}_1' = (E_1', E_1' u_1' \cos 60^\circ, E_1' u_1' \sin 60^\circ, 0).$$

Notice that we know very little about $\bar{p}_2'$, so it is better to write the equation of momentum

$$\bar{p}_2' = \bar{p}_1 + \bar{p}_2 - \bar{p}_1'.$$

Again, squaring gives

$$|\bar{p}_1|^2 + |\bar{p}_2|^2 + |\bar{p}_1'|^2 + 2\bar{p}_1 \cdot \bar{p}_2 - 2\bar{p}_1 \cdot \bar{p}_1' - 2\bar{p}_2 \cdot \bar{p}_1' = |\bar{p}_2|^2.$$

One has that

$$|\bar{p}_1|^2 = -m^2, \quad \bar{p}_1 \cdot \bar{p}_2 = -E_1 m_2, \quad \bar{p}_2 \cdot \bar{p}_1' = -m_2 E_1'.$$

Also,

$$\bar{p}_1 \cdot \bar{p}_1' = -E_1 E_1' + u_1 u_1' E_1 E_1' \cos 60^\circ.$$

Recall that $\cos 60^\circ = 1/2$. From all this it follows that

$$u_1 u_1' E_1 E_1' = 2E_1 E_1' + 2m_2 E_1' - 2E_1 m_2 - 2m_1^2.$$

Rearranging and factorising one obtains the desired result.

4. Before the disintegration one has

$$\bar{p} = m\gamma(1, v, 0, 0).$$

After

$$\bar{p}_{\gamma_1} = (E_1, E_1 \cos\alpha, E_1 \sin\alpha, 0), \quad \bar{p}_{\gamma_2} = (E_2, E_2 \cos\beta, -E_2 \sin\beta, 0).$$

Notice the minus sign in the last equation! Again, the conservation of momentum gives

$$\bar{p} = \bar{p}_{\gamma_1} + \bar{\gamma}_2.$$

Equating the components

$$E_1 \cos\alpha + E_2 \cos\beta = m_1 \gamma v,$$
$$E_1 \sin\alpha = E_2 \sin\beta,$$
$$E_1 + E_2 = m\gamma.$$

Substituting the second of these into the first and using the thrid to simplify one gets

$$E_1 \cos\alpha + E_1 \sin\alpha \cos\beta = (E_1 + E - 2)v = E_1 \left(1 + \frac{\sin\alpha}{\sin\beta}\right) v,$$

from where the result follows.