RELATIVITY (MTH6132)

SOLUTIONS TO THE PROBLEM SET 1

1. The vectorial Galilean transformation $\underline{r}' = \underline{r} - \underline{v}t$ implies

$$x' = x - v_x, \quad y' = y - v_y t, \quad z' = z - v_z t.$$

Also, in the Galilean framework one has that

$$t' = t.$$

The chain rule gives that

$$\frac{\partial}{\partial x} = \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial x}\frac{\partial}{\partial z'}.$$

From the Galilean transformations one obtains that

$$\frac{\partial t'}{\partial x} = 0, \quad \frac{\partial x'}{\partial x} = 1, \quad \frac{\partial y'}{\partial x} = 0, \quad \frac{\partial z'}{\partial x} = 0,$$

so that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}.$$

Similarly, one obtains that

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}.$$

The situation is different for the derivative with respect to the time. Again, the chain rule gives that

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial t}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial t}\frac{\partial}{\partial z'}, \\
&= \frac{\partial}{\partial t'} - v_x\frac{\partial}{\partial x'} - v_y\frac{\partial}{\partial y'} - v_z\frac{\partial}{\partial z'}, \\
&= \frac{\partial}{\partial t'} - \underline{v}\cdot\nabla'.
\end{aligned}$$

Substitution of these expressions into the original wave equation renders

$$\nabla'^2\phi = \frac{1}{c^2}\left(\frac{\partial^2\phi}{\partial t'^2} - 2\underline{v}\cdot\nabla'\frac{\partial\phi}{\partial t'} + (\underline{v}\cdot\nabla')^2\phi\right).$$

Comparing this expression with the original wave equation one sees that there are extra terms, and hence the equation is not invariant under Galilean transformations.

2. Starting from the Lorentz transformations in the form

$$t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad x' = \gamma\left(x - vt\right).$$

Adding and subtracting one obtains

$$ct' - x' = \varepsilon(ct - x), \quad ct' + x' = \varepsilon(ct + x),$$

with

$$\epsilon = \sqrt{\frac{1+v/c}{1-v/c}}.$$

To show that the composition of Lorentz transformations is a Lorentz transformation, assume that one has a frame F' moving with respect to F with speed v_1 and a frame F'' moving with respect to F' with speed v_2. One readily obtains that

$$ct' - x' = \sqrt{\frac{1+v_1/c}{1-v_1/c}}(ct - x),$$
$$ct'' - x'' = \sqrt{\frac{1+v_2/c}{1-v_2/c}}(ct' - x').$$

Combining these two expressions one obtains

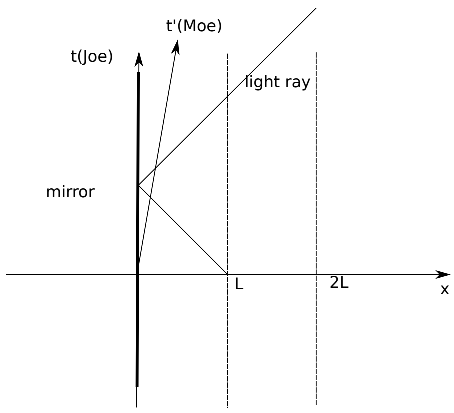
$$\begin{aligned} ct'' - x'' &= \sqrt{\frac{1+v_1/c}{1-v_1/c}}\sqrt{\frac{1+v_2/c}{1-v_2/c}}(ct - x), \\ &= \sqrt{\frac{1+\frac{v_1/c+v_2/c}{1+v_1v_2/c^2}}{1-\frac{v_1/c+v_2/c}{1+v_1v_2/c^2}}}(ct - x), \\ &= \sqrt{\frac{1+v/c}{1-v/c}}(ct - x), \end{aligned}$$

where

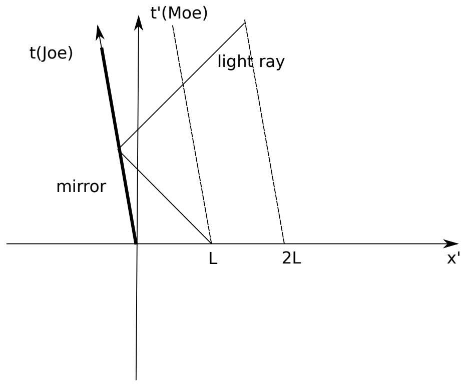
$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}.$$

This is of the required form and hence the composition of Lorentz transformations is a Lorentz transformation. This shows that the Lorentz transformations form a group.

3. The spacetime diagram from Joe's point of view is



From Moe's perspective one has


t'(Moe)
t(Joe)
light ray
mirror
L
2L
x'