Solution A1

A1(a) [*seen similar*]

The ”effective” gravitational mass of photon is

$$E/c^2 = h\nu/c^2. \tag{1}$$

From energy conservation

$$h\nu - \frac{Gm}{R}\frac{h\nu}{c^2} = h\nu_s - \frac{Gm}{R}\frac{h\nu_s}{c^2}, \tag{2}$$

where ν_s is the frequency of the photon at the surface of the planet.
Thus

$$\frac{\nu}{\nu_s} = \frac{1-\frac{Gm}{rc^2}}{1-\frac{Gm}{Rc^2}}. \tag{3}$$

[***? Marks***]

Taking into account that in Newtonian limit $Gm/rc^2 \ll 1$,
for redshift z we have

$$1+z = \frac{\lambda}{\lambda_s} = \frac{\nu_s}{\nu} \approx 1 - \frac{Gm}{Rc^2} + \frac{Gm}{rc^2}, \tag{4}$$

hence

$$z = \frac{Gm}{rc^2}\left(1-\frac{r}{R}\right) = \frac{Gm}{rc^2}(1-\frac{1}{3}) = \frac{2Gm}{3rc^2}. \tag{5}$$

Then taking into account that

$$m = \frac{4\pi\rho r^3}{3}, \quad r = \left(\frac{3m}{4\pi\rho}\right)^{1/3} \tag{6}$$

we obtain

$$z = \frac{2Gm}{3c^2}\left(\frac{3m}{4\pi\rho}\right)^{-1/3} = Am^{2/3}\rho^{1/3}, \text{ where } A = \frac{2G}{3c^2}(\frac{4\pi}{3})^{1/3}. \tag{7}$$

A1(b) [*seen similar*]

From Eq. (7)

$$m = Bz^{3/2}\rho^{-1/2}, \text{ where } B = \left(\frac{3c^2}{3G}\right)^{3/2}\left(\frac{3}{4\pi}\right)^{1/2}. \tag{8}$$

Then

$$m = \left(\frac{3\times(3\times10^{10}\mathrm{cm}\cdot\mathrm{s}^{-1})^2\times5\times10^{-10}}{2\times6.7\times10^{-8}\mathrm{cm}^3\mathrm{g}^{-1}\mathrm{s}^{-2}}\right)^{3/2}\left(\frac{3}{4\times3.14\times5\cdot\mathrm{g}\cdot\mathrm{cm}^{-3}}\right)^{1/2}\approx$$

$$\approx 6\times10^{27}\mathrm{g} = 6\times10^{24}\mathrm{kg} \tag{9}$$

(the mass of the Earth)

Solution A2

A2(a) [*seen similar*]

At the moment of the black hole formation the radius of the cloud is equal to its gravitational radius, r_g.

$$M = \frac{4\pi}{3}\rho_0 R_0^3, \text{ and } M = \frac{4\pi}{3}\rho_{BH} r_g^3. \tag{10}$$

Taking into account that

$$r_g = \frac{2GM}{c^2}, \quad \text{hence} \quad M = \frac{c^2}{2G} r_g, \tag{11}$$

we obtain

$$\frac{c^2}{2G} r_g = \frac{4\pi}{3}\rho_{BH} r_g^3, \quad \text{hence} \quad r_g = \sqrt{\frac{3c^2}{8\pi G\rho_{bh}}} = c\sqrt{\frac{3}{8\pi G\rho_{bh}}}. \tag{12}$$

A2(b) [*seen similar*]

Obviously

$$\rho_0 = \frac{\rho_{BH}}{\alpha} = 10^{4-9} \text{ kg/m}^{-3} = 1)^{-5} \text{ kg/m}^{-3}. \tag{13}$$

$$M = \frac{c^2}{2G} r_g = \frac{c^3}{2G}\sqrt{\frac{3}{8\pi G\rho_{bh}}} = \frac{1}{2}\sqrt{\frac{3}{8\pi}}\frac{c^3}{G^{3/2}\rho_{BH}^{1/2}} =$$

$$= \frac{27}{2\times 2\times 6.7}\sqrt{\frac{3}{2\pi\times 6.7}} \times 10^3 4 \text{ s}^{3-3}\text{m}^{3-3/2}\text{cm}^{-3/2}\text{kg}^{-1/2}\text{g}^{3/2} \approx$$

$$\approx 3\times 10^{33}\left(\frac{\text{m}}{\text{cm}}\right)^{3/2}\left(\frac{\text{g}}{\text{kg}}\right)^{3/2} \approx 3\times 10^{33}\left(10^{2-3}\right)\text{kg} \approx 10^{32}\text{kg}. \tag{14}$$

Solution A3

A3(a) [*seen similar*]

The Ergosphere is the region between the event horizon of rotating black hole and its limit of stationarity. The term ergosphere reflects the fact that it is possible to extract the rotational energy of the black hole, located outside the event horizon, with the help of some processes in ergosphere (like Penrose mechanism).
Thus, the inner radius of ergosphere is the radius of the event horizon, r_{EH}, which is determined from

$$g^{11} = 0, \ \text{ i.e. } \ g_{11} = \infty. \tag{15}$$

From the Kerr metric we obtain

$$\Delta \equiv r^2 - r_g r + a^2 = 0, \tag{16}$$

hence

$$r_{EH} = \frac{1}{2}\left(r_g + \sqrt{r_g^2 - 4a^2}\right), \ \text{ where } \ a = \frac{J}{Mc}. \tag{17}$$

The outer region of the ergosphere is the radius of the limit of stationarity, r_{LS}, which is determined from

$$g_{00} = 0. \tag{18}$$

From the Kerr metric we obtain

$$1 - \frac{r_g r}{\rho^2}, \ \text{ where } \ \rho^2 = r^2 + a^2 \cos^2\theta. \tag{19}$$

In the equatorial plane, where $\theta = \pi/2$, this gives

$$1 - \frac{r_g r}{r^2} = 0, \ \text{ hence } \ r_{LS} = r_g. \tag{20}$$

Hence,

$$f = \frac{r_{LS}}{r_{EH}} = \frac{2r_g}{r_g + \sqrt{r_g^2 - 4a^2}} = \frac{2}{1 + \sqrt{1 - \left(\frac{2a}{r_g}\right)^2}} =$$

$$= \frac{2}{1 + \sqrt{1 - \left(\frac{Jc}{GM^2}\right)^2}}. \tag{21}$$

A3(b) [*seen similar*]

From the previous equation

$$1 + \sqrt{1 - \left(\frac{Jc}{GM^2}\right)^2} = \frac{2}{f}, \tag{22}$$

$$J = \frac{GM^2}{c}\sqrt{1 - \left(\frac{2}{f} - 1\right)^2} = \frac{2GM^2}{cf}\sqrt{f - 1}. \tag{23}$$

From expression for r_{EH} we can see that

$$a \geq \frac{r_g}{2} = \frac{GM}{c^2}. \tag{24}$$

From the previous equation

$$a = \frac{r_g}{f}\sqrt{f-1}. \tag{25}$$

From Eq.? follows that

$$f = 2, \;\; if \;\; a = \frac{r_g}{2}. \tag{26}$$

It is easy to show that af is monotonic function in the range $1 \leq f \leq 2$. Indeed, in this range of f

$$\frac{da}{df} = \frac{r_g}{f^2}\left(\frac{f}{2\sqrt{f-1}} - \sqrt{f-1}\right) = \frac{r_g}{2f^2\sqrt{f-1}}\left(2-f\right) \geq 0. \tag{27}$$

Hence the rang of a corresponding to $f > 3/2$ is

$$r_g\frac{\sqrt{3/2-1}}{3/2} < a < \frac{r_g}{2}. \tag{28}$$

Finally

$$r_g\frac{\sqrt{2}}{3} < a < \frac{r_g}{2}. \tag{29}$$

Solution B1

B1(a) [*seen similar*]

To an order of magnitude gravitational force experienced by a particle of mass δm on the surface of the star from the star itself is $F_s \approx Gm\delta m/r^2$, while the tidal force producing a relative acceleration between the the same particle and the centre of the star to an order of magnitude is $F_{TD} \approx GM\delta m r/R^3$, hence defining the tidal radius as the radius at which $F_g \approx F_{TD}$, we have

$$\frac{Gm\delta m}{r^2} \approx \frac{GMr\delta m}{R_{TD}^3}, \tag{30}$$

and finally,

$$R_{TD} \approx r\left(\frac{M}{m}\right)^{1/3} \approx \left(\frac{3M}{4\pi\rho_s}\right)^{1/3}. \tag{31}$$

B1(b) [*seen similar*]

The critical mass, $M = M_{crit}$,can be found from the following equality

$$R_{TD} = r_g, \text{ where; } r_g = \frac{2GM_{crit}}{c^2}, \tag{32}$$

$$\left(\frac{3M_{crit}}{4\pi\rho_s}\right)^{1/3} = \frac{2GM_{crit}}{c^2}, \tag{33}$$

$$M_{crit}^{1/3}\rho_s^{-1/3} = AM_{crit}, \text{ where } A = \frac{24^{1/3}\pi^{1/3}G}{3^{1/3}c^2}, \tag{34}$$

$$\rho_s^{-1/3} = AM_{crit}^{2/3}, \text{ hence } M_{crit} = B\rho_s^{-1/2}, \text{ where } B = A^{-3/2} = \frac{3^{1/2}c^3}{42^{1/2}}\pi^{1/2}G^{3/2} \approx ? \tag{35}$$

$$\rho_s \approx \frac{3M_\odot}{4\pi 10^6 R_\odot^3}, \tag{36}$$

$$\text{hence, } M_{crit} \approx ? \tag{37}$$

B1(c) [*seen similar*]

c) Luminosity is

$$L \propto \int_{3r_g}^{R_{TD}} r'^2 dr' r'^{-2} \propto \int_{3r_g}^{R_{TD}} dr' \propto (R_{TD} - 3r_g) \propto\propto \left(AM^{1/3} - M\right), \text{ where } A = ?. \tag{38}$$

[***? Marks***] (*unseen*)

Taking into account that

$$R_{TD}(M_{crit}) = r_g(M_{crit}) = \frac{2GM_{crit}}{c^2}, \tag{39}$$

we have

$$L \sim x^{1/3} - Bx, \ \text{ where } \ x = M/M_{crit} \ \text{ and } \ B = ? \tag{40}$$

.

[? *Marks*]

From

$$\frac{dL}{dx} \propto \frac{1}{3}x^{-2/3} - B = 0, \ \text{ we have } \ x = (3B)^{-3/2} \approx ?. \tag{41}$$

Thus

$$M \approx ? M_{crit}. \tag{42}$$

B1(d) [*seen similar*]

Solution B2

B2(a) [*seen similar*]

Taking $\theta = \pi/2$ we can write down the Hamilton-Jacobi equation in the Schwarzschild metric as

$$\left(1-\frac{r_g}{r}\right)^{-1}\left(\frac{\partial S}{c\partial t}\right)^2 - \left(1-\frac{r_g}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2 - m^2c^2 = 0.$$

•**[3 Marks]**(*book work*)
Then putting $S = -Et + L\phi + S_r(r)$, we have for the radial component of the four-momentum
•**[4 Marks]**(*book work*)

$$\frac{\partial S}{\partial r} = p_1 = g_{11}p^1 = g_{11}\frac{dr}{ds} = \sqrt{\frac{E^2}{c^2}\left(1-\frac{r_g}{r}\right)^{-2} - \left(m^2c^2 + \frac{L^2}{r^2}\right)\left(1-\frac{r_g}{r}\right)^{-1}} =$$

$$= \frac{1}{c}\left(1-\frac{r_g}{r}\right)^{-1}\sqrt{E^2 - \left[mc^2\left(1+\frac{L^2}{m^2c^2r^2}\right)\left(1-\frac{r_g}{r}\right)\right]^2}.$$

•**[2 Marks]**(*book work*)
On other hand

$$\frac{dt}{ds} = p^0 = g^{00}p_0 = g^{00}\left(\frac{\partial S}{\partial t}\right) = -g^{00}E.$$

•**[6 Marks]**(*book work*)
Thus

$$\frac{dr}{dt} = \frac{\frac{dr}{ds}}{\frac{dt}{ds}} = \frac{1}{c}\left(1-\frac{r_g}{r}\right)\sqrt{E^2 - U_{eff}^2}\frac{1}{E} = \frac{1}{c}\left(1-\frac{r_g}{r}\right)^{-1}\sqrt{E^2 - U_{eff}^2},$$

where

$$U_{eff} = mc^2\sqrt{\left(1+\frac{L^2}{m^2c^2r^2}\right)\left(1-\frac{r_g}{r}\right)},$$

hence

$$E\left(1-\frac{r_g}{r}\right)^{-1}\frac{dr}{dt} = c\sqrt{E^2 - U_{eff}^2}.$$

Introducing $x = r_g/r$, we have $U_r^{'} = 0$ corresponds $U_x^{'} = 0$, so

$$[(1-x)(1+\alpha x^2]_x^{'} = 0, \tag{43}$$

where

$$\alpha = \frac{L^2}{m^2c^2r_g^2}, \tag{44}$$

$$-1 - 3\alpha x^2 + 2\alpha x = 0, \tag{45}$$

and

$$\alpha = \frac{1}{x(2-3x)}. \tag{46}$$

Then

$$\frac{E^2}{m^2c^4} = (1-x)(1+\frac{x}{2-3x}) = \frac{2(1-x)^2}{3-3x}, \tag{47}$$

and finally

$$E = \frac{\sqrt{2}mc^2(1 - r_g/r)}{(2 - 3r_g/r)^{1/2}} = \frac{\sqrt{2}mc^2(r - r_g)}{(2r - 3r_g)^{1/2}r^{1/2}}. \tag{48}$$

B2(b) [*seen similar*]

The effective potential energy includes potential energy and that part of kinetic energy, which is related with non-radial, angular motion. Points at which $E = U$, (E is the conservative total energy) correspond to turning points, where $dr/dt = 0$.

$$U = E, \;\; U^{'}_r = 0,$$

corresponds to the circular orbit, stable, if $U^{''}_{rr} > 0$, and unstable, if $U^{''}_{rr} < 0$.

B2(c) [*seen similar*]

The last circular orbit corresponds the following system of equations: $E = U$, $U^{'} = 0$, $U^{''} = 0$.

$$0 = U^{''} \sim 2\alpha(1 - 3x), \tag{49}$$

so $x = 1/3$, which corresponds to $r = 3r_g$.

$$\frac{E^2}{m^2c^4} = (1 - 1/3)(1 + 3/3^2) = 8/9, \tag{50}$$

and

$$E_{lo} = mc^2\frac{2\sqrt{2}}{3}. \tag{51}$$

B2(d) [*seen similar*]

Solution B3

B3(a) [*seen similar*]

The difference between Newtonian and general relativistic treatment is...?

The covariance principle says: The shape of all physical equations should be the same in an arbitrary frame of reference, including the most general case of non-inertial frames. If in contrast to the covariance principle the shape of physical equations were different in local inertial frames in presence of gravitational field and in non-inertial frames in absence of gravitational field then these equations would give different solutions, i.e. different predictions for (a) standing on the Earth, feeling the effects of gravity as a downward pull and (b) standing in a very smooth elevator that is accelerating upwards with the acceleration g , hence these equations would contradict to the basic postulate of the General Relativity, the principle of equivalence, which states that a uniform gravitational field (like that near the Earth) is equivalent to a uniform acceleration. Hence, the covariance principle is the mathematical formulation of the principle of equivalence.

B3(b) [*seen similar*]

$$DA_i = g_{ik} DA^k$$
$$DA_i = D(g_{ik} A^k) = g_{ik} DA^k + A^k D g_{ik},$$

hence

$$g_{ik} DA^k = g_{ik} DA^k + A^k D g_{ik},$$

which obviously means that

$$A^k D g_{ik} = 0.$$

Taking into account that A^k is arbitrary vector, we conclude that

$$D g_{ik} = 0.$$

Then taking into account that

$$D g_{ik} = g_{ik;m} dx^m = 0$$

for arbitrary infinitesimally small vector dx^m we have

$$g_{ik;m} = 0.$$

Introducing useful notation

$$\Gamma_{k,\, il} = g_{km} \Gamma^m_{il},$$

we have

$$g_{ik;\, l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk} \Gamma^m_{il} - g_{im} \Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,\, il} - \Gamma_{i,\, kl} = 0.$$

Permuting the indices i, k and l twice as

$$i \to k, \quad k \to l, \quad l \to i,$$

we have

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,\, il} + \Gamma_{i,\, kl}, \quad \frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,\, kl} + \Gamma_{l,\, ik} \quad \text{and} \quad -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,\, ki} - \Gamma_{k,\, li}.$$

Taking into account that

$$\Gamma_{k,\, il} = \Gamma_{k,\, li},$$

after summation of these three equation we have

$$g_{ik,l} + g_{li,k} - g_{kl,i} = 2\Gamma_{i,\, kl},$$

and finally

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right).$$

B3(c) [*seen similar*]

This situation corresponds to gravitational fields (for example, gravitational waves), when the space-time is curved, but matter is absent (empty space-time).

B3(d) [*seen similar*]

$$x_1 = r \cos \omega_0 t,$$

$$x_2 = r \sin \omega_0 t,$$

$$D_{11} = m r_c^2 (3 \cos^2 \omega_0 t - 1) = \frac{1}{2} m r^2 (1 + 3 \cos 2\omega_0 t),$$

$$D_{22} = m r_c^2 (3 \sin^2 \omega_0 t - 1) = \frac{1}{2} m r^2 (1 - 3 \cos 2\omega_0 t),$$

$$D_{12} = \frac{3}{2} m r_c^2 \sin 2\omega_0 t,$$

then

$$h_{11} = -\frac{2Gmr^2}{3c^4 R} \frac{3}{2} (2\omega_0)^2 \cos 2\omega_0 t = \frac{4\omega_0^2 G m r^2}{c^4 R} \cos 2\omega_0,$$

$$h_{22} = \frac{2Gmr^2}{3c^4 R} \frac{3}{2} (2\omega_0)^2 \cos 2\omega_0 t = -\frac{4\omega_0^2 G m r^2}{c^4 R} \sin 2\omega_0,$$

$$h_{12} = \frac{2Gmr^2}{3c^4 R} \frac{3}{2} (2\omega_0)^2 \sin 2\omega_0 t = \frac{4\omega_0^2 G m r^2}{c^4 R} \sin 2\omega_0,$$

it is clear, that

$$\omega = 2\omega_0.$$

From

$$r\omega_0^2 = \frac{GM}{r^2},$$

we have

$$\frac{1}{r^3} = \frac{\omega_0^2}{GM},$$

and finally

$$r_c^{-1} = (4GM)^{-1/3} \omega^{2/3}.$$

Thus

$$h \approx \frac{4\omega_0^2 G m r^2}{c^4 R} = \frac{r_g R_g}{r R} \approx \frac{r_g}{R} \left(\frac{R_g \omega}{c} \right)^{2/3}.$$