Relativistic Astrophysics. 2009. Course Work 6. Solutions

Q1.

a) A binary system consists of two neutron stars of the same mass M. The orbital period of the system is P. Using Newtonian mechanics, estimate to an order of magnitude the separation between the neutron stars, r, and the fractional relativistic corrections to the orbital motion.

Taking into account that

$$P = 2\pi/\omega,$$

where ω is the angular velocity which according Newtonian theory is related with the separation r as

$$\omega^2 r = \frac{GM}{r^2}, \quad \text{thus} \quad \omega = \sqrt{\frac{GM}{r^3}}.$$

Hence

$$r = \left(\frac{GM}{\omega^2}\right)^{1/3} = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3} = \left(\frac{T}{2\pi}\right)^{2/3} (GM)^{1/3}.$$

b) Evaluate the relativistic corrections if $P = 8$ min and $M = 1.5M_\odot$. Compare your estimate with relativistic effects in the solar system. It is known that the perihelion shift of Mercury is 43" per century. What analogous shift can you expect in the case of the binary system of neutron stars? (Hint: The relativistic shift per one orbital period is of order r_g/r, where r_g is gravitational radius of the neutron star.)

Per 1 year there are N revolutions and $N = \frac{1\ year}{P}$, hence the periastron shift per 1 year is

$$\Delta\varphi \sim \frac{r_g}{r} N \propto T^{-5/3}.$$

Comparing with the case of Mercury we have the ratio of the number of revolutions per year

$$N/N_{Merc} = P_{Merc}/P.$$

Hence

$$\frac{\Delta\varphi}{\Delta\varphi_{Merc}} \sim \left(\frac{M}{M_\odot}\right)\left(\frac{a}{r}\right)\left(\frac{P_{Merc}}{P}\right) = \left(\frac{M}{M_\odot}\right)\left(\frac{M}{M_\odot}\right)^{-1/3}\left(\frac{P}{P_{Merc}}\right)^{-2/3}\left(\frac{P_{Merc}}{P}\right) =$$

$$= \left(\frac{M}{M_\odot}\right)^{2/3}\left(\frac{P}{P_{Merc}}\right)^{-5/3}.$$

Taking to account that

$$P_{Merc} = (0.5)^{3/2}\ \text{year}, \quad \text{and} \quad P = \frac{5.7}{365}\ \text{year} \quad \approx \frac{1}{64}\ \text{year},$$

we obtain

$$\Delta\varphi = \left(\frac{43"}{100}\right) \times 10^{3\cdot 2/3}(0.5)^{3/2\cdot 5/3}(64)^{5/3} = \left(\frac{43}{60\times 60}\right)^o \times (0.5)^{5/2} \times 64^{5/3} =$$

$$\left(\frac{43 \times 0.25 \times 0.7 \times 4^5}{3600}\right)^o \approx (0.7/3.5)^o = 0.2^o.$$

Q2.

The quadrupole formula for the metric perturbation associated with gravitational waves is given by

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R}\frac{d^2 D_{\alpha\beta}}{dt^2}(t - R/c),$$

where R is the distance to the source of the gravitational waves and

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})dM$$

is the quadrupole tensor of the source. Consider a mass m moving along circular orbit around the black hole of mass M, assuming that $m \ll M$.

a) Show that all the amplitudes $h_{\alpha\beta}$ of gravitational wave, emitted by such system, are periodic functions of time with $\omega = 2\omega_0$, where $\omega_0 = 2\pi/T$, and T is the orbital period.

$$x_1 = r\cos\omega_0 t,$$

$$x_2 = r\sin\omega_0 t,$$

$$D_{11} = mr_c^2(3\cos^2\omega_0 t - 1) = \frac{1}{2}mr^2(1 + 3\cos 2\omega_0 t),$$

$$D_{22} = mr_c^2(3\sin^2\omega_0 t - 1) = \frac{1}{2}mr^2(1 - 3\cos 2\omega_0 t),$$

$$D_{12} = \frac{3}{2}mr_c^2\sin 2\omega_0 t,$$

then

$$h_{11} = -\frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\cos 2\omega_0 t = \frac{4\omega_0^2 Gmr^2}{c^4R}\cos 2\omega_0,$$

$$h_{22} = \frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\cos 2\omega_0 t = -\frac{4\omega_0^2 Gmr^2}{c^4R}\sin 2\omega_0,$$

$$h_{12} = \frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\sin 2\omega_0 t = \frac{4\omega_0^2 Gmr^2}{c^4R}\sin 2\omega_0,$$

it is clear, that

$$\omega = 2\omega_0.$$

b) Show that, to an order of magnitude (omitting the indices α and β)

$$h \approx \frac{r_g}{R}\left(\frac{R_g\omega}{c}\right)^{2/3},$$

where r_g is the gravitational radius of the mass m and R_g is the gravitational radius of the black hole.

From

$$r\omega_0^2 = \frac{GM}{r^2},$$

we have

$$\frac{1}{r^3} = \frac{\omega_0^2}{GM},$$

and finally

$$r_c^{-1} = (4GM)^{-1/3}\omega^{2/3}.$$

Thus

$$h \approx \frac{4\omega_0^2 Gmr^2}{c^4R} = \frac{r_g R_g}{rR} \approx \frac{r_g}{R}\left(\frac{R_g\omega}{c}\right)^{2/3}.$$

Q3.

The future LISA mission will be able to detect gravitational waves with $h > 10^{-23}$, if $10^{-4}Hz < \omega < 3 \cdot 10^{-3}Hz$. From what distance will it be possible to detect gravitational radiation from the binary system, containing the black hole of mass $m = 3M_\odot$, moving along a circular orbit with radius $r = 10^4 R_g$ around the massive black hole of mass $M = 10^3 M_\odot$?

$$\omega_0^2 = \frac{GM}{r^3} = \frac{+}{c^2} 2\frac{2GM}{c^2 r^3} = c^2 \frac{R_g}{2r^3},$$

hence,

$$\omega_0 = c\sqrt{\frac{R_g}{2r^3}} = c\sqrt{\frac{R_g}{2 \cdot 10^{12} R_g^3}} = \frac{10^{-6}c}{\sqrt{2}R_g} = \frac{10^{-4}Hz}{\sqrt{2}},$$

thus

$$\omega = 2\omega_0 = \sqrt{2}10^{-4}Hz \geq 10^{-4}Hz,$$

which means that the radiation is within LISA frequency range.

$$h = \frac{3 \cdot 10^5}{3 \cdot 10^{18}} \left(\frac{3 \cdot 10^5 \cdot 10^{-4}}{3 \cdot 10^{10}} \right)^{2/3} (\frac{m}{M.})(\frac{R}{1pc})^{-1}(\frac{M}{M.})^{2/3}(\frac{\omega}{10^{-4}Hz})^{2/3}$$
$$\approx 10^{-19}(\frac{m}{M.})(\frac{R}{1\;pc})^{-1}(\frac{M}{M.})^{2/3}(\frac{\omega}{10^{-4}Hz})^{2/3}.$$

Then

$$h = \frac{3 \cdot 10^5 cm}{R}(\frac{3 \cdot 10^5 \cdot 10^3 \cdot 1.4 \cdot 10^{-4} s^{-1} cm}{3 \cdot 10^{10}})^{2/3} > 10^{-23},$$

if

$$R < 3 \cdot 10^{23} \cdot 10^5 cm \cdot 10^{-4} \approx 1Mpc.$$