Relativistic Astrophysics. 2009. Course Work 5. Solutions

Q1.

A binary system consists of a invisible compact object of mass M_x and a visible star of mass M. The period of the orbit is T, the angle between the normal to the plane of the orbit to the line of sight of the observer is i and projection of orbital velocity of the visible star on the line of sight is v.

a) Which of values M_x, M, T, i and v can be determined from observations and how?

The observable values are T and v; The period T is measured by clocks, the velocity v is obtained from spectroscopic measurements based on Doppler effect.

b) Using Newtonian theory show that

$$\frac{(M_x \sin i)^3}{(M_x + M)^2} = \frac{v^3 T}{2\pi G}.$$

We need solve the following system if equations:

$$rM = r_x M_x,$$

$$\omega^2 r_x = GM(r_x + r)^{-2},$$
$$\omega^2 r = GM_x(r_x + r)^{-2},$$
$$v = \omega r \sin i.$$

Summing the second with the third, we have

$$\omega^2 (r_x + r) = GM(r_x + r)^{-2},$$

and

$$r_x + r = \left[\frac{G(M + M_x)}{\omega^2}\right]^{1/3},$$

$$r_x = r\frac{M}{M_x},$$

$$r(1 + \frac{M}{M_x}) = \left[\frac{G(M + M_x)}{\omega^2}\right]^{1/3},$$

$$v = \omega r \sin i = \omega \sin \frac{M_x}{M + M_x}\left[\frac{G(M + M_x)}{\omega^2}\right]^{1/3},$$
$$= (G\omega)^{1/3} \sin i \frac{M_x}{(M + M_x)^{2/3}},$$
$$\frac{v^3}{G\omega} = \frac{v^3 T}{2\pi G} = \frac{M_x^3 \sin^3 i}{(M_x + M)^2}.$$

c) Assume that observations of a binary system give strong evidence, that the visible star is periodically eclipsed by the invisible object (eclipsed binary). What can you say in this case about the orientation of the binary system?

In the case we can say, that the line of sight is very close to the orbital plane, which means that $\sin i \approx 1$.

d) Imagine that you obtained several sets of observations for eclipsed binaries. Results of these observations are summarized below:

Object N	1	2	3	4	5
Velocity in km/s	200	300	500	1000	2000
Period in min	5	10	20	50	100

Assume that any invisible compact object is a black hole, if its mass exceeds $3M_\odot$. Assume also that the mass of all visible stars in your binaries have masses between $M_\odot$ and $5M_\odot$. Which binary system contains, may contain and does not contain a black hole?

For all these objects $\sin i \approx 1$.
Introducing

$$m_x = M_x/M_\odot,$$
$$m = M/M_\odot$$

and

$$f = \frac{m_x^3}{(m_x+m)^2}$$

we have:

for $m_x = 3$ and $m = 1$

$$f = \frac{27}{16} \approx 1.7,$$

for $m_x = 3$ and $m = 5$

$$f = \frac{27}{64} \approx 0.42.$$

Thus:

if $f < 0.42$ there is no black hole,
if $0.42 < f < 1.7$ there may be a black hole,
if $1.7 < f$ there is a black hole.

Taking into account that according to Q1(b)

$$f = \left(\frac{v^3 T}{2\pi G M_\odot}\right) = \left(\frac{v}{100\text{km s}^{-1}}\right)^3 \left(\frac{T}{5\text{min}}\right) \left(\frac{10^6 \cdot 300}{\pi c^2 \frac{2GM_\odot}{c^2}}\right) \text{km}^3\text{s}^{-2} =$$

$$= \left(\frac{v}{100\text{km s}^{-1}}\right)^3 \left(\frac{T}{5\text{ min}}\right) \left(\frac{10^{-2}}{3\pi r_{g\odot}}\right) \text{km} = 3,5\cdot 10^{-4} \left(\frac{v}{100\text{km s}^{-1}}\right)^3 \left(\frac{T}{5\text{ min}}\right).$$

Object 1: $f \approx 0.0028$.
Object 2: $f \approx 0.019$.
Object 3: $f \approx 0.177$.
Object 4: $f \approx 3.54$.
Object 5: $f \approx 56.62$.

Thus, the final answer is:
the objects N 4 and 5 contain black holes, the object N 1 - 3 do not contain black holes.

Q2.

The "effective potential energy" is defined as

$$U(r) = mc^2 \left(1 - \frac{r_g}{r}\right)^{1/2} \left(1 + \frac{L^2}{m^2c^2r^2}\right)^{1/2},$$

where L is the angular momentum and m is the mass of a particle, moving around Schwarzschild black hole.

a) What is the physical meaning of the "effective potential energy"? Explain how using U to find stable and unstable circular orbits.

The effective potential energy includes potential energy and that part of kinetic energy, which is related with non-radial, angular motion. Points at which $E = U$, (E is the conservative total energy) correspond to turning points, where $dr/dt = 0$.

$$U = E, \;\; U_r^{'} = 0,$$

corresponds to the circular orbit, stable, if $U_{rr}^{''} > 0$, and unstable, if $U_{rr}^{''} < 0$.

b) Using the Hamilton-Jacobi equation, show that the energy of a particle moving along circular orbit depends on the radius of the orbit as follows

$$E(r) = \sqrt{2}mc^2 \frac{(r - r_g)}{(2 - 3r_g)^{1/2}\, r^{1/2}}.$$

Introducing $x = r_g/r$, we have $U_r^{'} = 0$ corresponds $U_x^{'} = 0$, so

$$[(1 - x)(1 + \alpha x^2]_x^{'} = 0,$$

where

$$\alpha = \frac{L^2}{m^2c^2r_g^2},$$

$$-1 - 3\alpha x^2 + 2\alpha x = 0,$$

and

$$\alpha = \frac{1}{x(2 - 3x)}.$$

Then

$$\frac{E^2}{m^2c^4} = (1 - x)(1 + \frac{x}{2 - 3x}) = \frac{2(1 - x)^2}{3 - 3x},$$

and finally

$$E = \frac{\sqrt{2}mc^2(1 - r_g/r)}{(2 - 3r_g/r)^{1/2}} = \frac{\sqrt{2}mc^2(r - r_g)}{(2r - 3r_g)^{1/2}r^{1/2}}.$$

c) Determine the radius of the last circular orbit. What fraction of the initial energy will be released by the particle when it reaches the last circular orbit?

The last circular orbit corresponds the following system of equations: $E = U$, $U^{'} = 0$, $U^{''} = 0$.

$$0 = U^{''} \sim 2\alpha(1 - 3x),$$

so $x = 1/3$, which corresponds to $r = 3r_g$.

$$\frac{E^2}{m^2c^4} = (1 - 1/3)(1 + 3/3^2) = 8/9,$$

and

$$E_{lo} = mc^2 \frac{2\sqrt{2}}{3}.$$

Fraction of energy:

$$f = \frac{E_\infty - E_{lo}}{E_\infty} = 1 - \frac{2\sqrt{2}}{3} = 0.057$$

Q3.

a) Using the definition of the wave-vector for a photon, $k^i = \frac{dx^i}{d\lambda}$, where λ is an arbitrary scalar parameter along the world-line, derive the Eikonal equation and explain how the solution of this equation is related to k^i.

From the definition of the four-vector for light

$$k^i = \frac{dx^i}{d\lambda}$$

we have

$$ds^2 = g_{ik}dx^i dx^k = g_{ik}k^i k^k d\lambda^2.$$

Then taking into account that

$$ds = 0$$

we have

$$k_i k^i = g^{ik}k_i k_k = 0.$$

Substituting covariant vector

$$k_i = -\frac{\partial \psi}{\partial x^i},$$

we obtain the Eikonal Equation in gravitational field

$$g^{ik}\frac{\partial \Psi}{\partial x^i}\frac{\partial \Psi}{\partial x^k} = 0.$$

b) Consider the propagation of the photon in the equatorial plane of a Schwarzshild black hole. Given that the solution of the Eikonal equation can be written in the form

$$\Psi = -\omega t + \frac{\omega \rho}{c}\phi + \Psi_r(r),$$

where ω is the frequency of the photon and ρ is its impact parameter, find a differential equation for Ψ_r and show that

$$\frac{dr}{dt} = c(1 - \frac{r_g}{r})\sqrt{1 - \frac{\rho^2}{r^2} + \frac{r_g \rho^2}{r^3}}.$$

Substituting the above expression for Ψ to the Eikonal equation we obtain

$$\frac{1}{1-\frac{r_g}{r}}\frac{\omega^2}{c^2}-\frac{1}{r^2}\left(\frac{b\omega}{c}\right)^2-\left(1-\frac{r_g}{r}\right)(-k_1)^2=0,$$

where

$$k_1\equiv k_r=-\Psi_{,1}=-\frac{d\Phi_r(r)}{dr},$$

hence

$$k_1=\pm\sqrt{\frac{1}{1-\frac{r_g}{r}}\left[\frac{1}{1-\frac{r_g}{r}}\frac{\omega^2}{c^2}-\frac{b^2\omega^2}{c^2r^2}\right]}.$$

Taking account that

$$\frac{dr}{dt}=\frac{\frac{dr}{d\lambda}}{\frac{dt}{d\lambda}}=\frac{k^1}{k^0}=\frac{g^{11}k_1}{g^{00}k_0}=-\frac{cg_{00}k_1}{g_{11}\omega},$$

we can easy find that

$$\frac{dr}{dt}=c(1-\frac{r_g}{r})\sqrt{1-\frac{\rho^2}{r^2}+\frac{r_g\rho^2}{r^3}}.$$

c) Sketch the regions of possible propagation in the $(\tilde{r},\tilde{\rho})$ diagram, where $\tilde{r}=r/r_g$ and $\tilde{\rho}=\rho/r_g$.

See Fig. 5.4.

d) Show that photons can propagate along a circular orbit and find the radius of this orbit $\tilde{r}_c$ and corresponding impact parameter $\tilde{\rho}_c$. Show that this orbit is unstable.

From

$$U_{eff}=1,\quad \frac{dU_{eff}}{dr}=0,$$

where

$$U_{eff}=\frac{b^2}{r^2}\left(1-\frac{r_g}{r}\right)\quad\text{and}\quad\rho\equiv b$$

we obtain

$$\frac{b^2}{r^2}\left(1-\frac{r_g}{r}\right)=1$$

and

$$-\frac{2b^2}{r^3}\left(1-\frac{r_g}{r}\right)+\frac{b^2r_g}{r^4}=0,$$

hence

$$-2\left(1-\frac{r_g}{r}\right)+\frac{r_g}{r}=0$$

and finally

$$r=\frac{3}{2}r_g.$$