Course Work 5

Q1.

A binary system consists of a invisible compact object of mass M_x and a visible star of mass M. The period of the orbit is T, the angle between the normal to the plane of the orbit to the line of sight of the observer is i and projection of orbital velocity of the visible star on the line of sight is v.

a) Which of values M_x, M, T, i and v can be determined from observations and how?

b) Using Newtonian theory show that

$$\frac{(M_x \sin i)^3}{(M_x + M)^2} = \frac{v^3 T}{2\pi G}.$$

c) Assume that observations of a binary system give strong evidence, that the visible star is periodically eclipsed by the invisible object (eclipsed binary). What can you say in this case about the orientation of the binary system?

d) Imagine that you obtained several sets of observations for eclipsed binaries. Results of these observations are summarized below:

Object N	1	2	3	4	5
Velocity in km/s	200	300	500	1000	2000
Period in min	5	10	20	50	100

Assume that any invisible compact object is a black hole, if its mass exceeds $3M_\odot$. Assume also that the mass of all visible stars in your binaries have masses between $M_\odot$ and $5M_\odot$. Which binary system contains, may contain and does not contain a black hole?

Q2.

The "effective potential energy" is defined as

$$U(r) = mc^2 \left(1 - \frac{r_g}{r}\right)^{1/2} \left(1 + \frac{L^2}{m^2 c^2 r^2}\right)^{1/2},$$

where L is the angular momentum and m is the mass of a particle, moving around Schwarzschild black hole.

a) What is the physical meaning of the "effective potential energy"? Explain how using U to find stable and unstable circular orbits.

b) Using the Hamilton-Jacobi equation, show that the energy of a particle moving along circular orbit depends on the radius of the orbit as follows

$$E(r) = \sqrt{2}mc^2 \frac{(r - r_g)}{(2r - 3r_g)^{1/2}\, r^{1/2}}.$$

c) Determine the radius of the last circular orbit. What fraction of the initial energy will be released by the particle when it reaches the last circular orbit?

Q3.

a) Using the definition of the wave-vector for a photon, $k^i = \frac{dx^i}{d\lambda}$, where λ is an arbitrary scalar parameter along the world-line, derive the Eikonal equation and explain how the solution of this equation is related to k^i.

b) Consider the propagation of the photon in the equatorial plane of a Schwarzshild black hole. Given that the solution of the Eikonal equation can be written in the form

$$\Psi = -\omega t + \frac{\omega\rho}{c}\phi + \Psi_r(r),$$

where ω is the frequency of the photon and ρ is its impact parameter, find a differential equation for Ψ_r and show that

$$\frac{dr}{dt} = c(1 - \frac{r_g}{r})\sqrt{1 - \frac{\rho^2}{r^2} + \frac{r_g\rho^2}{r^3}}\ .$$

c) Sketch the regions of possible propagation in the $(\tilde{r}, \tilde{\rho})$ diagram, where $\tilde{r} = r/r_g$ and $\tilde{\rho} = \rho/r_g$.

d) Show that photons can propagate along a circular orbit and find the radius of this orbit $\tilde{r}_c$ and corresponding impact parameter $\tilde{\rho}_c$. Show that this orbit is unstable.