Course Work 4. Solutions

Q1.

Write a brief essay (1-2 Pages) "Why invisible black holes can be seen in astronomical observations".

Black holes can be detected only by their gravitational interaction with matter and electromagnetic waves outside the event horizon. Although black holes cannot be detected directly, many observational studies have provided substantial evidence for black holes. Stellar mass black holes have masses that are equivalent to the masses of individual stars 415 times the mass of our Sun. Our Milky Way galaxy contains several probable stellar-mass black holes. These candidates are all members of X-ray binary systems in which the denser object draws matter from its partner via an accretion disk. A supermassive black hole is a black hole with a mass of an order of magnitude between 10^5 and 10^{10}. It is currently thought that almost all galaxies, including the Milky Way, contain supermassive black holes at their galactic centers. There is also evidence that two supermassive black holes can co-exist in the same galaxy for a certain amount of time.

Q2.

Explain briefly what is the main difference between the limit of stationarity and the event horizon of a black hole?

The Limit of stationarity (Static Limit): the interval ds for test particle in rest

$$dr = d\theta = d\phi = 0.$$

In this case

$$ds^2 = g_{00} dx^{0^2},$$

We can see that if $g_{00} = 0$, then $ds^2 = 0$, which means that the world line of particle in rest is the world line of light. Hence, at the surface $g_{00} = 0$ no particle with finite rest mass can be in rest. For this reason this surface is called the limit of stationarity.

Event Horizon is a spherically symmetric surface

$$F(r) = \text{const}.$$

Its normal vector is defined as usually as

$$n_i = F_{,i} = \delta_i^1 \frac{dF}{dr}.$$

If at this surface $g^{11} = 0$ then

$$g^{ik} n_i n_k = g^{11} n_1 n_1 = g^{11} \left(\frac{dF}{dr}\right)^2 = 0,$$

which means that n_i is a null vector and any particle with finite rest mass can not move outward the surface $g^{11} = 0$, thus this surface is the event horizon.

Q3.

Given that n_i is a covariant vector and

$$n^i = g^{ik} n_k.$$

Taking into account that g_{ik} and g^{ik} are reciprocal to each other, show that

$$g^{ik} n_i n_k = g_{ik} n^i n^k.$$

$$g^{ik} n_i n_k = g^{ik} g_{im} n^m g_{kp} n^p = \delta^k_m n^m g_{kp} n^p = g_{kp} n^m n^p = g_{ik} n^i n^k.$$

Q4.

Let the interval ds is given as

$$ds^2 = \left(1 - \frac{A}{x^1}\right) (dx^0)^2 - \left(1 - \frac{4A^2}{(x^1)^2}\right)^{-1} (dx^1)^2 - (dx^2)^2 - (dx^3)^2,$$

where A is a constant.

a) Find g^{11}.

$$g^{11} = \frac{g_{00} g_{22} g_{33}}{g_{00} g_{11} g_{22} g_{33}} = \frac{1}{g_{11}} = -\left(1 - \frac{4A^2}{(x^1)^2}\right).$$

b) Show that this interval corresponds to a space-time geometry with the limit of stationarity and the event horizon. Determine the position of both these surfaces.

The Limit of stationarity corresponds to

$$g_{00} = \left(1 - \frac{A}{x^1}\right) = 0,$$

hence $x^1 = A$ is the limit of stationarity;
Event Horizon corresponds to a spherically symmetric surface

$$g^{11} = -\left(1 - \frac{4A^2}{(x^1)^2}\right) = 0,$$

hence $x^1 = 2A$ is the event horizon.

Q5.

a) Discuss briefly what is the significant deference between the "Laplacian black hole" and the black hole in General Relativity.

In the case of "Laplacian black hole" a body with the velocity less than velocity of light, at the beginning moves outward and only after some time starts to move inward, while in the case of black hole in General Relativity motions outward are impossible, because the surface $r = r_g$ is the event horizon.

b) Explain why the surface $r = r_g$ in the Schwarzschild metric is the event horizon. Where the limit of stationarity is located?. Show that the surface $r = r_g$ is a null surface.

$$F(r) = r - r_g = 0$$

is obviously a spherically symmetric surface and its normal vector is

$$n_i = F_{,i} = \delta_i^1 \frac{dF}{dr} = \delta_i^1$$

At this surface

$$g^{11} = \frac{1}{g_{11}} = 1 - \frac{r_g}{r} = 1 - \frac{r_g}{r_g} = 0,$$

hence

$$g^{ik} n_i n_k = g^{11} n_1 n_1 = g^{11} = 0,$$

which means that n_i is a null vector and any particle with finite rest mass can not move outward the surface $g^{11} = 0$, thus this surface is the event horizon.
The limit of stationarity is located at the surface

$$g_{00} = 1 - \frac{r_g}{r} = 0,$$

which in the Schwarzschild metric is located also at the surface

$$r = r_g.$$

This surface is called null surface because the world line

$$r = r_g, \; d\theta = 0, \; d\phi = 0,$$

obviously belong to the surface

$$F(r) = r - r_g = 0$$

and corresponds to a world line of a photon moving outward with the speed of light.

Q6.

Consider a rotating black hole described by the Kerr metric.

a) What is meant by the event horizon, the "limit of stationarity" and the "ergosphere"? (Compare with the case of the Schwarzschild black hole).

The Kerr metric describing the gravitational field of rotating bodies has the following form:

$$ds^2 = (1 - \frac{r_g r}{\rho^2})dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2 + \frac{2 r_g r a}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the specific angular momentum.
For the Kerr metric $g_{00} = 0$ gives

$$1 - \frac{r_g r}{\rho^2} = 0,$$

thus

$$r^2 - r_g r + a^2\cos^2\theta = 0,$$
$$\Delta = r^2 - r_g r + a^2 = 0,$$

and

$$r_{st} = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2\cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2\cos^2\theta}.$$

The location of horizon in the Kerr metric: $g^{11} = 0$ ($g_{11} = \infty$) corresponds to

$$\Delta = r^2 - r_g r + a^2 = 0,$$

and

$$r = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2 \cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2 \cos^2\theta}.$$

$$r_{hor} = \frac{r_g}{2} \pm \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}.$$

One can see easily that

$$r_{st} \geq r_{hor},$$

for example,

$$r_{st} = r_{hor}, \text{ if } \theta = 0, \text{ or } \theta = \pi \text{ (at the poles)},$$

and

$$r_{st} = 2r_g > r_{hor}, \text{ if } \theta = \frac{\pi}{2} \text{ (at the equator)}.$$

The region between the limit of stationarity and the event horizon is called the "ergosphere".

In the Schwarzschild metric as was shown in the previous question

$$r_{hor} = r_{st},$$

which means that in this case the "ergosphere" does not exist.

b) Describe briefly the Penrose process of extraction of energy from a rotating black hole and explain why this mechanism does not contradict to the statement, that nothing can escape from within black hole.

By the Penrose mechanism it is possible to extract rotational energy of Kerr black hole. That extraction is made possible because the rotational energy of the black hole is located not inside the event horizon, but outside in a curl gravitational field. Such field is also called gravimagnetic field. All objects in the ergosphere are unavoidably dragged by the rotating spacetime. The Penrose mechanism: Some body enters into the "ergosphere" and decays then into two pieces. The momentum of the two pieces of matter can be arranged so that one piece escapes to infinity, whilst the other falls past the outer event horizon into the black hole. The escaping piece of matter can have greater mass-energy than the original infalling piece of matter.

Q7.

a) Give the definition of the Ricci tensor R_{ik} and prove that

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}$$

By definition the Ricci tensor is

$$R_{ik} = g^{lm} R_{limk} = R^l_{ilk},$$

where the curvature Riemann tensor is defined by

$$A_{i;\,k;\,l} - A_{i;\,l;\,k} = A_m R^m_{ikl}.$$

By straightforward calculations

$$A_{i;k;l} - A_{i;l;k} = A_{i;k,l} - \Gamma^m_{li} A_{m;k} - \Gamma^m_{lk} A_{i;m} - A_{i;l,k} + \Gamma^m_{ki} A_{m;l} + \Gamma^m_{kl} A_{i;m} =$$

$$\left(A_{i,k} - \Gamma^m_{ik} A_m\right)_{,l} - \Gamma^m_{li}\left(A_{m,k} - \Gamma^n_{mk} A_n\right) - \left(A_{i,l} - \Gamma^m_{il} A_m\right)_{,k} + \Gamma^m_{ki}\left(A_{m,l} - \Gamma^n_{ml} A_n\right) =$$

$$A_{i,k,l} - A_{i,l,k} - \Gamma^m_{ik} A_{m,l} - \Gamma^m_{il} A_{m,k} - \Gamma^m_{kl} A_{i,m} + \Gamma^m_{il} A_{m,k} + \Gamma^m_{ik} A_{m,l} + \Gamma^m_{lk} A_{i,m} -$$

$$-\Gamma^m_{ik,l} A_m + \Gamma^m_{il}\Gamma^p_{mk} A_p + \Gamma^m_{kl}\Gamma^p_{im} A_p +$$

$$+\Gamma^m_{ik,l} A_m - \Gamma^m_{ik}\Gamma^p_{ml} A_p - \Gamma^m_{lk}\Gamma^p_{im} A_p =$$

$$= A_m\left(-\Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} + \Gamma^p_{kl}\Gamma^m_{ip} + \Gamma^m_{il,k} - \Gamma^p_{ik}\Gamma^m_{pl} - \Gamma^p_{lk}\Gamma^m_{ip}\right) = A_m\left(-\Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} + \Gamma^m_{il,k} - \Gamma^p_{ik}\Gamma^m_{pl}\right),$$

hence

$$R^m_{ikl} = \Gamma^m_{il,k} - \Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} - \Gamma^p_{ik}\Gamma^m_{pl},$$

and replacing k by l and l by k and then just putting $m = l$ we finally obtain

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

b) Starting from the Einstein equations in the form

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik},$$

where G is the gravitational constant, prove that

$$T^i_k = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right).$$

Contracting with g^{ik}, we have the Einstein equations in mixed form

$$R^i_k = \frac{8\pi G}{c^4}(T^i_k - \frac{1}{2}\delta^i_k T).$$

$$R = g^{ik} R_{ik} = \frac{8\pi G}{c^4}\left(g^{ik} T_{ik} - \frac{1}{2} g^{ik} g_{ik} T\right) = \frac{8\pi G}{c^4}\left(T^i_i - \frac{1}{2}\delta^i_i T\right) = \frac{8\pi G}{c^4}\left(T - \frac{1}{2} 4\right) = -\frac{8\pi G}{c^4} T.$$

Thus

$$T = -\frac{c^4}{8\pi G} R.$$

Thus

$$T_{ik} = \frac{c^4}{8\pi G}\left(R_{ik} - \frac{1}{2} g_{ik} R\right),$$

then in mixed form we have

$$T^i_k = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right).$$

c) What can you say about the nature of gravitational field, for which $R_{ik} = 0$, while R_{ikln} is not equal to zero?

This situation corresponds to gravitational fields (for example, gravitational waves), when the space-time is curved, but matter is absent (empty space-time).