Relativistic Astrophysics. 2009. Course Work 3. Solutions

Q1.

a) Explain what is the reciprocal tensor.

Two tensors A_{ik} and B^{ik} are called reciprocal to each other if

$$A_{ik}B^{kl} = \delta_i^l.$$

b) Demonstrate how using the reciprocal contravariant metric tensor g^{ik} and the covariant metric tensor g_{ik} you can form contravariant tensor from covariant tensors and vice versa.

We can introduce a contravariant metric tensor g^{ik} which is reciprocal to the covariant metric tensor g_{ik}:

$$g_{ik}g^{kl} = \delta_i^l.$$

With the help of the metric tensor and its reciprocal we can form contravariant tensor from covariant tensors and vice versa, for example:

$$A^i = g^{ik}A_k, \quad A_i = g_{ik}A^k.$$

c) Show that in an arbitrary non-inertial frame

$$g^{ik} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3},$$

where $S^i_{(0)k}$ is the transformation matrix from locally inertial frame of reference (galilean frame) to this non-inertial frame.

We know that in the galilean frame of reference

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \eta^{ik} \equiv \mathrm{diag}(1, -1, -1, -1),$$

hence

$$g^{ik} = S^i_{(0)n}S^k_{(0)m}\eta^{lm} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3}.$$

Q2.

a) Give a rigorous proof that the interval is a scalar.

Given that g_{ik} is a covariant tensor of the second rank and that

$$ds^2 = g_{ik} dx^i dx^k,$$

hence,

$$ds^2 = g_{ik} dx^i dx^k = (\tilde{S}_i^n \tilde{S}_k^m g'_{nm})(S_p^i dx'^p)(S_w^k dx'^w) = (\tilde{S}_i^n S_p^i)(\tilde{S}_k^m S_w^k)(g'_{nm} dx'^p dx'^w) =$$

$$= \delta_p^n \delta_w^m (g'_{nm} dx'^p dx'^w) = g'_{pw} dx'^p dx'^w = g'_{ik} dx'^i dx'^k = ds'^2,$$

thus

$$ds = ds'$$

which means that ds is a scalar.

b) Prove that the metric tensor is symmetric.

$$ds^2 = g_{ik} dx^i dx^k = \frac{1}{2}(g_{ik} dx^i dx^k + g_{ik} dx^i dx^k) = \frac{1}{2}(g_{ki} dx^k dx^i + g_{ik} dx^i dx^k) = \frac{1}{2}(g_{ki} + g_{ik}) dx^i dx^k =$$

$$= \tilde{g}_{ik} dx^i dx^k,$$

where

$$\tilde{g}_{ik} = \frac{1}{2}(g_{ki} + g_{ik}),$$

which is obviously symmetric one. Then we just drop ”˜”.

Q3.

Using lecture notes 3, write a short essay (1-2 pages) "Proper time and physical distances".

Proper time: the world line of an observer who uses some clock to measure the proper time, $d\tau$, between two infinitesimally close events in the same place in space is

$$dx^1 = dx^2 = dx^3.$$

Defining proper time exactly as in Special Relativity:

$$d\tau = \frac{ds}{c},$$

we have

$$ds^2 \equiv c^2 d\tau^2 = g_{ik} dx^i dx^k = g_{00}(dx^0)^2,$$

thus

$$d\tau = \frac{1}{c}\sqrt{g_{00}}dx^0.$$

For the proper time between any two events occurring at the same point in space we have

$$\tau = \frac{1}{c}\int \sqrt{g_{00}}dx^0.$$

Spatial distance: separating the space and time coordinates in ds we have

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0 dx^\alpha + g_{00}(dx^0)^2.$$

To define dl we will use a light signal according to the following procedure: From some point B with spatial coordinates $x^\alpha + dx^\alpha$ a light signal emitted at the moment corresponding to time coordinate $x^0 + dx^{0(1)}$ propagates to a point A with spatial coordinates x^α and then after reflection at the moment corresponding to time coordinate x^0 the signal propagates back over the same path and is detected in the point B at the moment corresponding to time coordinate $x^0 + dx^{0(2)}$ as shown below. The interval between the events which belong to the same world line of light in Special and General Relativity is always equal to zero:

$$ds = 0.$$

Solving this equation with respect to dx^0 we find two roots:

$$dx^{0(1)} = \frac{1}{g_{00}}\left(-g_{0\alpha}dx^\alpha - \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}\right)$$

$$dx^{0(2)} = \frac{1}{g_{00}}\left(-g_{0\alpha}dx^\alpha + \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}\right)$$

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}}\sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}.$$

Then

$$dl = \frac{c}{2}d\tau = \frac{c}{2}\frac{\sqrt{g_{00}}}{c}(dx^{0(2)} - dx^{0(1)})$$

and finally

$$dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta, \text{ where } \gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}.$$

Q4.

a) Show that all covariant derivatives of metric tensor are equal to zero.

$$DA_i = g_{ik}DA^k$$
$$DA_i = D(g_{ik}A^k) = g_{ik}DA^k + A^k Dg_{ik},$$

hence

$$g_{ik}DA^k = g_{ik}DA^k + A^k Dg_{ik},$$

which obviously means that

$$A^k Dg_{ik} = 0.$$

Taking into account that A^k is arbitrary vector, we conclude that

$$Dg_{ik} = 0.$$

Then taking into account that

$$Dg_{ik} = g_{ik;m}dx^m = 0$$

for arbitrary infinitesimally small vector dx^m we have

$$g_{ik;m} = 0.$$

b) Find the the relationship between the Cristoffel symbols and first partial derivative of the metric tensor.

Introducing useful notation

$$\Gamma_{k,\, il} = g_{km}\Gamma^m_{il},$$

we have

$$g_{ik;\,l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,\, il} - \Gamma_{i,\, kl} = 0.$$

Permuting the indices i, k and l twice as

$$i \to k, \quad k \to l, \quad l \to i,$$

we have

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,\, il} + \Gamma_{i,\, kl}, \quad \frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,\, kl} + \Gamma_{l,\, ik} \text{ and } -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,\, ki} - \Gamma_{k,\, li}.$$

Taking into account that

$$\Gamma_{k,\, il} = \Gamma_{k,\, li},$$

after summation of these three equation we have

$$g_{ik,l} + g_{li,k} - g_{kl,i} = 2\Gamma_{i,\, kl},$$

and finally

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right).$$