# Relativistic Astrophysics. 2009. Course Work 1. Solutions

## Q1.

***a) Following the original Laplace calculation within the framework of Newtonian gravity, show that the escape velocity from the surface of a gravitating body of mass $M$ is equal to the speed of light, if the radius of the body is equal to $2GM/c^2$ (gravitational radius).***

The escape velocity from the surface of this body of mass $M$ and of radius $R$ is

$$v_{esc} = \sqrt{\frac{2GM}{R}}.$$

The escape velocity is equal to the speed of light

$$v_{esc} = c$$

if

$$R = r_g \equiv \frac{2GM}{c^2}.$$

***b) A star forms a black hole of mass $M$. Show that to an order of magnitude its density at the moment immediately before the formation of the black hole is***

$$2 \times 10^{16}\ g \cdot cm^{-3} \left(\frac{M}{M_\odot}\right)^{-2}.$$

***For what mass is this equal to the density of air ($\approx 10^{-5} g/cm^3$)?***

To an order of magnitude

$$\rho = \frac{M}{V} = \frac{M}{\frac{4\pi}{3} r_g^3} = \frac{3M}{4\pi\left(\frac{2GM}{c^2}\right)^3} = \frac{3M_\odot}{4\pi\left(\frac{2GM_\odot}{c^2}\right)^3}\left(\frac{M}{M_\odot}\right)^{-2} =$$

$$= \frac{3}{4\pi}\frac{M_\odot}{(3\ \mathrm{km})^3}\left(\frac{M}{M_\odot}\right)^{-2} = \frac{3 \cdot 2 \times 10^{30}\ \mathrm{kg}}{4 \cdot 3.14 \cdot 3^3 \times 10^9\ \mathrm{m}^3}\left(\frac{M}{M_\odot}\right)^{-2} \approx 2 \times 10^{19} \mathrm{kg} \cdot \mathrm{m}^{-3}\left(\frac{M}{M_\odot}\right)^{-2}.$$

If this density is equal to the density of air, then

$$\rho = 10^{-5} g/cm^3 = \frac{10^{-5} \cdot 10^{-3}\mathrm{kg}}{10^{-6}\mathrm{m}^3} = 10^{-2}\mathrm{kg} \cdot \mathrm{m}^{-3},$$

hence

$$10^{-2}\mathrm{kg} \cdot \mathrm{m}^{-3} = 2 \times 10^{19}\mathrm{kg} \cdot \mathrm{m}^{-3}\left(\frac{M}{M_\odot}\right)^{-2}$$

and

$$\frac{M}{M_\odot} \approx \left(2 \cdot 10^{21}\right)^{1/2} \approx 4 \cdot 10^{10}.$$

The final answer is

$$M = 4 \cdot 10^{10} M_\odot.$$

## Q2.

***a) Consider a photon with energy $E = h\nu$ climbing out of a gravitational field and use energy conservation law to show that in traveling through a potential difference $\delta U \ll c^2$, the photon should experience a fractional frequency shift***

$$z = -\frac{\delta U}{c^2}.$$

An effective gravitational mass of a photon of frequency of $\nu$ is

$$m_{photo} = \frac{E_{photon}}{c^2} = \frac{h\nu}{c^2}.$$

From conservation of energy we have

$$h\nu_{ob} - \frac{Gm}{R_{ob}}\frac{h\nu_{ob}}{c^2} = h\nu_{em} - \frac{Gm}{R_{em}}\frac{h\nu_{em}}{c^2},$$

where "ob"corresponds to observation and "em" to emission of the photon. Thus

$$\frac{\nu_{ob}}{\nu_{em}} = \frac{1 - \frac{Gm}{R_{em}c^2}}{1 - \frac{Gm}{R_{ob}c^2}} \approx 1 - \frac{Gm}{R_{em}c^2} + \frac{Gm}{R_{ob}c^2}.$$

Taking into account that in Newtonian limit

$$\delta U = \frac{Gm}{R_{ob}} - \frac{Gm}{R_{em}} < 0,$$

we have

$$z = \frac{|\nu_{obs} - \nu_{em}|}{\nu_{em}} = 1 - \left[1 - \frac{Gm}{R_{em}c^2} + \frac{Gm}{R_{ob}c^2}\right] = -\frac{\delta U}{c^2}.$$

***b) From observations of some unknown object it was found that a fractional frequency shift of spectral lines was equal to***

$$z = \frac{|\delta\nu|}{\nu} = (3.2 \pm 1.2)\cdot 10^{-4}.$$

***Assuming that this redshift is explained by the redshift in the gravitational field of a solar mass object, calculate the predictable redshifts caused by a star of solar type, a white dwarf and a neutron star, whose typical radii may be taken to be 700000 km, 6000 km and 10 km, respectively. Hence determine which type of object was observed.***

For a star of solar type

$$z \approx \frac{GM}{rc^2} = \frac{r_{g\odot}}{2r}\left(\frac{M}{M\odot}\right) \approx \frac{3}{2\cdot 7\cdot 10^5} \approx 2\cdot 10^{-6};$$

For a white dwarf

$$z \approx \frac{GM}{rc^2} = \frac{r_{g\odot}}{2r}\left(\frac{M}{M\odot}\right) \approx \frac{3}{2\cdot 6\cdot 10^3} \approx 2.5\cdot 10^{-4};$$

For a neutron star

$$z \approx \frac{GM}{rc^2} = \frac{r_{g\odot}}{2r}\left(\frac{M}{M\odot}\right) \approx \frac{3}{2\cdot 10} \approx 0.15.$$

The final answer is: a white dwarf was observed.

**Q3**.

***a) Using simple Newtonian estimates, show that the radius of tidal disruption, $R_{TD}$, for a star of mass $m$ and radius $r$ in the gravitational field of the black hole of mass $M$ is***

$$R_{TD} \approx r \left(\frac{M}{m}\right)^{1/3}.$$

Self-gravity force is

$$F_{sg} \approx Gm\delta m/r^2.$$

The tidal force is

$$F_{TD} \approx GM\delta mr/R^3.$$

The tidal radius is determined from

$$F_{sg} \approx F_{TD},$$

hence

$$\frac{Gm\delta m}{r^2} \approx \frac{GM\delta mr}{R_{TD}^3}$$

and, finally,

$$R_{TD} \approx r(M/m)^{1/3}.$$

***b) Find the critical value of the black hole mass, $M_{crit}$, for which $R_{TD}$ equals the gravitational radius of the black hole.***

The tidal radius for the solar type star is equal to the gravitational radius if

$$R_{TD} = r_\odot \left(\frac{M}{M_\odot}\right)^{1/3} = r_g = \frac{2GM}{c^2} = 3\ km \frac{M}{M_\odot},$$

hence

$$r_\odot \left(\frac{M_{cr}}{M_\odot}\right)^{1/3} = r_{g\odot} \left(\frac{M_{cr}}{M_\odot}\right),$$

hence

$$\left(\frac{M_{cr}}{M_\odot}\right)^{2/3} \approx \left(\frac{r_\odot}{r_{g\odot}}\right)$$

and, finally,

$$M_{cr} \approx M_\odot \left(\frac{r_\odot}{r_{g\odot}}\right)^{3/2} \approx \left(\frac{7 \cdot 10^5}{3}\right)^{3/2} \approx 10^8 M_\odot.$$