Quantum Physics – Homework 9

Due Thursday 22nd March at 4. Attempt answers to all questions.

Hand in your script by the deadline into the post box near the secretaries' offices on level 1. **Assignments handed in past the deadline will not be marked.** Course title, week number and student name should appear on every sheet of the worked exercises, which should be securely bound together. Please *also* write your tutor's initials and time of tutorial on the cover sheet.

Problem 1 `[30 marks]`

A particle of mass m is free to move in the one-dimensional box defined by $0 \leq x \leq L$. At the time $t = 0$ its quantum state is described by the wavefunction

$$\psi(x) \;=\; N\sqrt{\frac{2}{L}}\left(\sin\frac{\pi x}{L} + i\sqrt{3}\sin\frac{3\pi x}{L}\right) \;.$$

(i) Rewrite the wavefunction in terms of stationary states and determine the normalisation constant N such that the wavefunction is normalised. **[6]**

(ii) Calculate the expectation value of the position of the particle. **[6]**

(iii) What are the possible outcomes of a measurement of the energy of the particle, and what the corresponding probabilities? **[5]**

(iv) Write the wavefunction of the particle for $t > 0$. **[5]**

(v) Determine the probability of finding the particle in the left half of the box. **[8]**

Problem 2 `[20 marks]`

Consider a particle which can move in an infinite square well between $x = 0$ and $x = L$. The particle is in a state described by the wavefunction $\psi^{(n)}(x)$ corresponding to a state with definite energy $E_n = \hbar^2\pi^2 n^2/(2mL^2)$, where n is a given integer.

(i) Calculate $\langle x\rangle$ and $\langle p\rangle$ in this state, defined as

$$\langle x\rangle \;=\; \int_0^L dx\; x\,\psi^*(x)\psi(x) \;,$$

and

$$\langle p \rangle \;=\; -i\hbar \int_0^L dx\ \psi^*(x)\frac{d}{dx}\psi(x)\ .$$

[4]

(ii) Calculate $\langle x^2 \rangle$ in this state. You may find the following integral useful:

$$\int_0^{n\pi} dz\ z^2 \sin^2 z \;=\; \frac{\pi n}{12}(-3 + 2n^2\pi^2)\ ,$$

where n is an integer. **[5]**

(iii) Determine $\langle p^2 \rangle$. This quantity is defined as

$$\langle p^2 \rangle \;=\; \int_0^L dx\ \psi^*(x)\Big(-\hbar^2 \frac{d^2}{dx^2}\Big)\psi(x)\ .$$

Hint: Notice that

$$\frac{d^2}{dx^2}\psi^{(n)}(x) = -\left(\frac{\pi n}{L}\right)^2 \psi^{(n)}(x)\ .$$

[6]

(iv) Finally determine $\Delta x \Delta p$ in this state, and check that it satisfies Heisenberg's uncertainty principle

$$\Delta x \Delta p \geq \hbar/2\ .$$

Recall that $\Delta x := (\langle x^2 \rangle - \langle x \rangle^2)^{1/2}$ and $\Delta p := (\langle p^2 \rangle - \langle p \rangle^2)^{1/2}$. **[5]**

Problem 3 (optional)

The expectation value of the momentum of a particle moving in the one dimension parameterised by x in the quantum state given by the wavefunction ψ is defined by

$$\langle p \rangle_\psi \;:=\; \int_{\mathcal{D}} dx\, \psi^*(x)\Big(-i\hbar \frac{d}{dx}\Big)\, \psi(x)\ ,$$

where $\mathcal{D}$ is the domain where the particle can move. Consider the case where $\mathcal{D}$ is the line interval $a \leq x \leq b$. Furthermore, consider the particular case where $\psi(x)$ is a *real* wavefunction, i.e. $\psi^*(x) = \psi(x)$. Prove that $\langle p \rangle_\psi = 0$ for any normalisable real wavefunction.