2B27 Problem Sheet 3

1. Show that the maximum power of the wind blowing with velocity v at an angle θ with respect to the normal of a wind turbine, rotor diameter d can expressed as

$$P = \frac{1}{8}\rho\pi d^2 v^3 \cos^3\theta.$$

[5]

Briefly explain why the maximum obtainable power from the wind on a turbine which is 100% efficient must be less than this. **[3]**
Write down an expression for the coefficient of performance and show that it has a maximum of 16/27. **[2]**

Solution

Consider wind velocity v incident at angle θ with respect to the normal of a disc, area A. The wind velocity normal to the disc will be

$$v_\perp = v\cos\theta.$$

So the kinetic energy carried by a mass or air, dm miving through A in time dt will be

$$dE = \frac{1}{2} dm v^2 \cos^2\theta$$

but $dm = \rho A dl$ where $dl = vdt$ is the distance traveled by the wind in time dt, so

$$dE = \frac{1}{2}\rho A v\cos\theta dt v^2\cos^2\theta.$$

But $A = \pi r^2 = \pi d^2/4$ so

$$P \equiv \frac{dE}{dt} = \frac{1}{8}\rho\pi d^2 v^3\cos^3\theta.$$

For power to be extracted from the air, the air must be able to get away from the turbine, or else there will be no flow. The air must therefore still have some kinetic energy after passing through the turbine, so it is not possible to extract all the kinetic energy carried by the wind.
The maximum amount of energy that can be extracted with a turbine is know as the Betz limit. If the coefficient of performance C_F is given by

$$C_F = 4\varepsilon(1-\varepsilon)^2$$

then this has a maximum when $dC_F/d\varepsilon = 0$, ie

$$4(1-\varepsilon)^2 - 8\varepsilon(1-\varepsilon) = 0.$$

$$(1-\varepsilon) - 2\varepsilon = 0 \quad \Rightarrow \quad \varepsilon = \frac{1}{3}.$$

Which gives us the maximum value, $C_F = \frac{16}{27}$.

2. An estuary has area 100 km^2, width 0.5 km and depth at low tide 25 m. If the tidal range is 6 m, calculate the average power that could be obtained from the estuary using a barrage scheme. **[5]**

Calculate the average flow rate during the rising tide. **[3]**

Calculate the power that could be obtained from the average rising tidal flow using a 100% efficient underwater turbine with blade radius 8 m. Why would the average power during the rising tide be higher than this? **[6]**

How many turbines would be required to power a small town requiring 20 MW. **[1]**

Solution

The estuary has surface area $A = 100 \times 10^6$ m^2 and width 500 m, and tidal range 6 m at high tide. Therefore, the extra volume in the estuary at high tide is $V = Ah$ and the potential energy of this additional mass of water (assuming all the mass is at the common centre of gravity) is given by

$$\begin{aligned} E = mg\frac{h}{2} &= \frac{1}{2}Ah\rho gh \\ &= \frac{1}{2}A\rho gh^2 \\ &= \frac{1}{2} \times 10^8 \times 9.81 \times 1027 \times 36 \\ &= 1.813 \times 10^{13}\text{J}. \end{aligned}$$

The tidal period is $T = 12$ hours 25 minutes $= 44700$ s, so the average power is

$$P = E/T = \frac{1.813 \times 10^{13}}{44700} = 4.056 \times 10^8\text{W} = 405.6\text{MW}.$$

The average volume flow rate, is the total volume over the time it flows, ie the water flows in, and out during one tidal cycle, so during the rising tide) ie time $T/2$, the flow rate is

$$\begin{aligned} F &= \frac{V}{T/2} \\ &= \frac{2Ah}{T} \\ &= 1.68 \times 10^4\text{m}^3\text{s}^{-1} \end{aligned}$$

The mass flow rate (ie the above $\times$ the density of sea water, 1027 kg m^{-3}) is also acceptable. During the rising tide, the water flows through an aperture area $a =$width$\times$hight. At high tide, the height is the depth at low tide (ie 25 m) + the tidal range, ie 31 m. At low tide, it is 25m. Assuming the average of 28 m then

$$a = 500 \times 28\text{m}$$

so the velocity of flow, is the volume flow rate through the apperture, ie

$$\begin{aligned} v &= \frac{2Ah}{Ta} \\ &= \frac{2 \times 10^8 \times 6}{44700 \times 500 \times 28} \\ &= 1.92\text{ms}^{-1}. \end{aligned}$$

So for a turbine which was 100% efficient (ie the Betz limit) then the maximum power is

$$\begin{aligned}
P &= \frac{1}{2} C_F \rho A v^3 \\
&= \frac{1}{2} \times \frac{16}{27} \times 1027\pi \times 64 \times 1.92^3 \\
&= 4.33 \times 10^5 \mathrm{W} \\
&= 433 \mathrm{kW}.
\end{aligned}$$

The average power would most likely be more than this, since this is the power calculated at the average flow rate. Since the power goes as v^3, so higher velocities would contribute more, ie

$$\bar{P} \sim \frac{\int v^3 dt}{\int dt} > \left[\frac{\int v dt}{\int dt} \right]^3.$$

For a town requiring 20 MW, then a total of

$$n = \frac{20}{0.433} = 46.2$$

ie 47 turbines would be required if power was generated during both the rising and falling tides, or 94 if power was generated only during the rising, or during the falling tide.

3. A small block of flats has side lengths 10 m and 5 m and the rooms on each floor are 2 m high. All the flats are kept at the same temperature so heat loss through the ceiling and floor of the middle flat can be neglected. If the overall U value for the walls of the flats is 2 W $m^{-2}K^{-1}$ and the input and output boundary layer resistances h are 0.1 K m^2W^{-1} calculate the steady state temperature of the middle flat if the flat is heated by a 1 kW heater and the outside temperature is 0° C. **[5]**

A building with surface area A and volume V has an overall thermal transmittance (including boundary resistance) of U. The outside temperature varies with time t as

$$T_{\text{out}} = T_0 \exp(-kt)$$

where T_0 is a constant. Derive the following equation determining the evolution of the temperature, T, inside the building.

$$T + \frac{\rho V c_v}{AU}\frac{dT}{dt} = T_0 \exp(-kt).$$

[4]

Solve this differential equation to show that the temperature can be expressed as

$$T = \frac{T_0}{AU - k\rho V c_v}\left[AU\exp(-kt) - k\rho V c_v \exp\left(\frac{-AUt}{\rho V c_v}\right)\right].$$

[11]

Solution

The U value including the thermal resistances of the boundary layers is given by

$$\begin{aligned}
\frac{1}{U} &= \frac{1}{U_{\text{walls}}} + r_{\text{in}} + r_{\text{out}} \\
&= \frac{1}{2} + 0.1 + 0.1 \\
&= 0.7 \\
\Rightarrow U &= 1.428\text{Wm}^{-2}\text{K}^{-1}.
\end{aligned}$$

The surface area of the walls of the middle flat is

$$A = 2 \times (10 \times 2) + 2 \times (5 \times 2) = 60\text{m}^2.$$

For a system in thermal equilibrium with heat input rate dI/dt, heat loss rate dQ/dt and rate of internal energy change dU/dt then

$$\frac{dI}{dt} = \frac{dQ}{dt} + \frac{dU}{dt}.$$

So in the steady state case, where $dI/dt = 10^3$ W and the internal energy is not changing, then $dU/dt = 0$. Since $dQ/dt = AU\Delta T$ then

$$\begin{aligned}
\frac{dI}{dt} &= AU\Delta T \\
\Delta T &= \frac{1}{AU}\frac{dI}{dt} \\
&= \frac{10^3}{60 \times 1/0.7} \\
&= 11.7°\text{C}.
\end{aligned}$$

Since the outside temperature is $0°$ C, so the temperature of the room is

$$T = 11.7°\text{C}.$$

Consider a building volume, V and surface area A with outside temperature, T_{out} with time dependence

$$T_{\text{out}} = T_0 \exp -kt.$$

Since

$$\frac{dI}{dt} = \frac{dQ}{dt} + \frac{dU}{dt}$$

Then if there is no heating, $dI/dt = 0$ so

$$\frac{dQ}{dt} = AU\Delta T \quad \text{and} \quad \frac{dU}{dt} = mc_v\frac{dT}{dt} = \rho V c_v \frac{dT}{dt}$$

where m is the mass of the air and c_v is the heat capacity at constant volume (naïvely asuming no volume dependence on temperature). Therefore

$$AU\Delta T + \rho V c_v \frac{dT}{dt} = 0.$$

but $\Delta T = T - T_{\text{out}}$ so

$$T - T_0 \exp -kt + \frac{\rho V c_v}{AU}\frac{dT}{dt} = 0$$
$$\Rightarrow T + \frac{\rho V c_v}{AU}\frac{dT}{dt} = T_0 \exp -kt.$$

To solve this for T, we first solve the auxilliary equation

$$T + \frac{\rho V c_v}{AU}\frac{dT}{dt} = 0$$

since you can always add a solution to this equation, to get the general solution, so we try a form

$$T = \mathcal{A} \exp -\kappa t.$$

Substituting into our auxilliary equation we get

$$\mathcal{A} \exp -\kappa t - \frac{\kappa \rho V c_v}{AU}\mathcal{A} \exp -\kappa t = 0$$

$$\Rightarrow 1 - \frac{\kappa \rho V c_v}{AU} = 0$$
$$\Rightarrow \kappa = \frac{AU}{\rho V c_v}$$

which gives us

$$T = \mathcal{A} \exp \frac{-AUt}{\rho V c_v}.$$

Now we need to find a particular integral to our original equation. We try a solution of the same form as the right hand side of our equation

$$
\begin{aligned}
T &= \mathcal{B} \exp -kt \\
\Rightarrow \mathcal{B} \exp -kt - k\mathcal{B}\frac{rhoVc_v}{AU} \exp -kt &= T_0 \exp -kt \\
\Rightarrow \mathcal{B}\left[1 - \frac{k\rho V c_v}{AU}\right] = T_0 \Rightarrow \mathcal{B} &= \frac{AUT_0}{AU - k\rho V c_v}.
\end{aligned}
$$

So our overall solution becomes

$$
T = \mathcal{A} \exp \frac{-AUt}{\rho V c_v} + \left[\frac{AUT_0}{AU - k\rho V c_v}\right] \exp -kt.
$$

Now we take the boundary condition $T(t = 0) = T_0$, so that

$$
T_0 = \mathcal{A} + \frac{AUT_0}{AU - k\rho V c_v}
$$

so that

$$
\begin{aligned}
\mathcal{A} &= T_0 - \frac{AUT_0}{AU - k\rho V c_v} \\
&= \frac{AUT_0 - T_0 k\rho V c_v - AUT_0}{AU - k\rho V c_v} \\
&= \frac{-T_0 k\rho V c_v}{AU - k\rho V c_v}
\end{aligned}
$$

So that our overall solution becomes

$$
T = \frac{T_0}{AU - k\rho V c_v}\left[-k\rho V c_v \exp\left(\frac{-AUt}{\rho V c_v}\right) + AU \exp -kt\right]
$$

NB. In the original problem sheet, there were some typos, in that the - was missing in the coefficient of the first exponent and a factor of $-t$ was missing in the first exponent. Allowances have been made for this in the marking.